Estimating the Spectral Risk Measure with Distorted Functions over Normal and Uncertainty Market Events

Estimación de medida espectral de riesgo con funciones distorsionadas en eventos de mercados normales e inciertos

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Abstract

Market risk models such as Value at Risk (VaR) and Conditional Value at Risk (CVaR), have demonstrated significant limitations in predicting losses during financial crises, including the COVID-19 pandemic. While these models perform adequately under normal market conditions, they fail to assign appropriate weights to losses based on their magnitude, particularly during extreme events. This study proposes a robust framework for implementing a spectral risk measure (SRM) based on distortion functions, aiming to address the shortcomings of traditional methods in both normal and extreme market environments. By incorporating distortion functions, the spectral risk measure assigns weights to losses proportionally to their size, making it adaptable to different investor profiles, including both risk-averse or inexperienced agents and experienced investors. This approach provides a more accurate assessment of market risk, optimizes portfolio management, enhances risk estimation processes, refines financial and banking risk management practices, and strengthens the overall framework for market risk measurement.

Keywords: Value at risk; spectral measure of risk; probability; distortion function; market risk; estimation.

JEL classification: G17, C13, C15, C12.

Resumen

Modelos como el VaR y el CVaR han demostrado no ser eficaces en predecir pérdidas en crisis, como la pandemia de covid-19. Aunque son coherentes bajo condiciones normales de mercado, no asignan peso a las pérdidas por su tamaño, especialmente en eventos extremos. Este estudio propone un enfoque robusto en la implementación de una medida espectral de riesgo basada en funciones de distorsión para mejorar la estimación de pérdidas en eventos normales y extremos de mercado, superando las limitaciones de mediciones tradicionales. Al integrar funciones de distorsión, se tiene una medida espectral que asigna peso a las pérdidas por el tamaño de esta, y se ajusta a perfiles aversos o inexpertos, así como para aquellos experimentados, midiendo el riesgo de mercado correctamente. Este enfoque optimiza la gestión de portafolios, mejora la estimación de riesgos, afina las prácticas financieras y bancarias, y robustece la medición de riesgo de mercado.

Palabras clave: valor en riesgo; medida espectral de riesgo; probabilidad; función de distorsión; riesgo de mercado; estimación.

Códigos JEL: G17, C13, C15, C12.

Introduction

Over the past three decades, academic and empirical literature on market risk measurement for financial institutions operating in increasingly uncertain environments has grown significantly. Prominent models such as Value at Risk (VaR) and Conditional Value at Risk (CVaR) have received extensive attention. However, studies by Nupur & Bhabani (2020) and Muneer & Lakshmi (2022) demonstrate that these models fail to effectively predict losses during crises, such as the COVID-19 pandemic. Similarly, both Wang (2002) and Dowd (2005) argued that VaR and CVaR do not fully utilize the available information in return distributions.

Theoretical developments in risk measurement have their roots in the proposals of Artzner et al. (1998, 1999) on coherent risk measures, emphasizing properties such as subadditivity. These developments were extended by Delbaen (2000) who introduced probability spaces in the context of coherent market risk measures, formalizing the VaR concept. Although CVaR was highlighted by Acerbi & Tasche (2002) for meeting axiomatic properties, studies such as those by Dowd (2005), argue that both VaR and CVaR fail to adequately exploit the information contained in return distributions, particularly in tail events. Building on these critiques, definitions of spectral risk measures were established by Acerbi & Tasche (2002) reinforcing the need for more sophisticated models. Dowd (2005) further developed the spectral risk measure (SRM) as a weighting function tied to the agent's risk profile, and later Dowd *et al*. (2008) integrated utility functions into SRM frameworks, observing limitations in their applicability across diverse risk profiles.

Recent empirical contributions also highlight the need for new approaches. Research by Ho & Nguyen (2017) on S&P500 and VNIndex portfolios demonstrated the utility of asymmetric distortion functions for risk measurement, while Brandtner (2018) analyzed how SRMs can influence risk aversion incentives based on the agent's profile. During the COVID-19 pandemic, studies such as those by Nupur & Bhabani (2020) revealed an increase in market correlations, and Muneer & Lakshmi (2022) highlighted the effectiveness of the Kupiec test over traditional models in evaluating VaR.

This study proposes a spectral risk measure based on distortion functions capable of integrating the agent's risk profile and providing a more robust risk assessment under highly uncertain events, overcoming the limitations in predicting maximum losses during adverse market events of traditional models, such as historical simulation and parameterization. Additionally, this approach is relevant for optimizing risk measurement, which is crucial for capital management and the solvency of financial institutions. The document is divided into five sections: introduction and literature review; theoretical foundation and proposed model; methodology and application; results; and, finally, conclusions and potential extensions.

1. Value at risk (VaR), definitions and generalities

A probability space is defined, where market information is considered, -algebra , which can be generated by a limited collection of random variables. Additionally, a random variable is defined by compiling the profits and losses associated with the assets traded in this market, and a variable containing the set of all risks. A risk measure is then defined as a function:

1.1 Value at risk

1.1.1 Traditional VaR or VaR by percentiles T3

Delbaen (2000) applies probability spaces to coherent measures of market risk, defining a random variable $X \text{ y } 0 < \alpha < 1$; then q is said to be α – th *cuantil* if $\mathbb{P}[X < q] \leq \alpha \leq \mathbb{P}[X \leq q]$ defining the VaR expression as follows:

$$\rho = VaR_{\alpha =} - q_{\alpha}^{+}(X) \tag{1}$$

1.1.2 VaR by historical simulation

Based on historical data (JPMorgan/Reuters, 1996), portfolios are valued over time windows (six months to two years) without assumptions regarding return distributions. Dowd (2005) defines this methodology as taking the observation from the loss tail, which is part of ordered losses and gains series (P&L).

1.1.3 Parametric VaR

The methodology is incorporated based on the technical framework used by Risk Metrics JPMorgan/Reuters (1996) and estimates the maximum potential change in a portfolio, assuming returns with a normal or log-normal distribution with a given probability over a time horizon. Volatility and correlations are modeled using methods such as the Exponential Weighted Moving Average (EWMA) or the Generalized Autoregressive Conditional Heteroscedasticity (GARCH) method to capture market events:

$$\rho = \alpha VaR = P_{t-1} - P^* = P_{t-1}(1 - exp[\mu_R - \sigma_R Z_\alpha]) \quad (2)$$

Where P_{t-1} is a loss given in period $t-1$, P^* is the loss equivalent to VaR, Z_α is a standard normal variable corresponding to the confidence level α and where logarithmic returns are assumed to be normally distributed with mean μ_R and standard deviation σ_R.

1.1.4 Conditional VaR or expected shortfall

Acerbi & Tasche (2002) define the tail expected loss, also known as Conditional VaR (CVaR) or Expected Shortfall (ES), as the expected loss if a loss exceeds the VaR value or the average of the worst loss scenarios, where is the chosen confidence level:

$$\rho = ES = \frac{1}{1-\alpha}\int_{\alpha}^{1} VaR_u du \quad (3)$$

CVaR or ES complies with the axiomatic framework, provides a better description of extreme losses, and contributes to maximizing expected utility. However, this measure fails to assign greater weight to extreme losses in the tail of the distribution (Dowd, 2005).

1.2 Coherent risk measures

Artzner et al., (1998) claim that for a risk measure to be coherent it must meet the following properties which must be satisfied by all these types of measures:

Sub-additivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

Homogeneity: $\rho(tX) = t\rho(X)$.

Monotonicity: $\rho(X) \geq \rho(Y)\ if\ X \leq Y$.

Translation Invariance: $\rho(X+n) = \rho(X) - n$.

1.2.1 Spectral risk measure (SRM)

Acerbi (2007) provides a general formula and defines it as "the weighted average of all portfolio outcomes from the worst (p=0) to the best (p=1)", where the residual freedom is tied to the choice of weight for the function ϕ. Thus, the spectral measure is defined as:

$$\rho_{\phi}(X) = -\int_0^1 \phi(p) F_X^{-1}(p)\, dp \tag{4}$$

Where the risk spectrum is the function $\phi\colon [0,1] \mapsto R$ and satisfies the following conditions: 1) Non-negativity: $\phi(p) \geq 0,\ 2)$, 2) Normalization: $\int_0^1 \phi(p)dp = 1$ and 3) Monotonicity: $\phi(p_1) \geq \phi(p_2)\ if\ p_1 \geq p_2$.

Additionally, Dowd (2005) highlights that ES corresponds to a special case of the SRM, which assigns the same weight to the loss tail beyond the confidence level, while assigning a zero weight to the rest of the quantiles as demonstrated. On the contrary, it is expected that "a risk measure assigns a greater weight to larger losses, or at least equal weight to smaller losses" Dowd (2005). Both Dowd (2005) and Acerbi (2007) emphasize the importance of designing adaptive measures that properly weigh extreme losses and propose spectral and distortion functions since they offer a robust framework for integrating probabilistic information and investor preferences, overcoming traditional VaR and ES limitations.

1.3 Distorted risk measures

Yaari (1987) laid the foundation for distorted risk measures by developing the theory of function transformation. Subsequently, Wang (2002) proposed an innovative framework using the distortion operator, which employs Choquet's integral to value under risk neutrality and address issues associated with distorted risk measures. This operator enables the transformation of probability distributions to calculate risk-adjusted premiums in the insurance context, ensuring resulting functions exhibit monotonic growth, concavity, and valid probabilities that adhere to first- and second-order stochastic dominance rules, thereby demonstrating the equivalence between transformation and distortion methods.

1.3.1 Wang's distortion function

Wang's distortion function, also known as the distortion operator, is based on the standard normal distribution and is defined as:

$$g_{\alpha}(u) = \Phi[\Phi^{-1}(u) - \alpha] \tag{5}$$

Where α is a real-valued parameter, Φ(*x*) is defined as the cumulative standard normal distribution, and $u = \Phi(x)$ is the inverse function of $x = \Phi^{-1}(u)$. This operator is notable for: 1) its ability to adjust systematic risk (α) in assets, correlating it with risk-adjusted returns, and 2) its use in evaluating coherent risk measures. In this regard, Wang (2002) shows that traditional VaR overlooks the size of losses, while CVaR, although it improves this aspect, ignores losses outside of specific quantiles.

1.4 Relationship with spectral measures and extended properties

Authors such as Gzyl & Mayoral (2008) connect distorted and spectral measures, demonstrating that the latter inherit key properties from the former, such as concavity and coherence. Likewise, Balbás et al. (2008) introduce advanced concepts such as completeness and exhaustiveness, ensuring that distortion functions use all the information from the initial distribution and preserve stochastic dominance.

Meanwhile, Balbás et al. (2008) analyze the properties of distortion measures and reveal the distorted risk measure as the expectation of a new variable where probabilities have been reassigned, and which, by the properties of Choquet's integral, satisfies all the properties defined for coherent risk measures.

1.5 Spectral risk measure and distorted functions

Ho & Nguyen (2017) use three distortion functions, which integrate the spectral risk measure as follows:

Dual Power Function:

$$M_\phi = \int_0^1 \gamma p^{\gamma-1} q_x(p)dp \tag{6}$$

Proportional Hazard Risk Transformation Function:

$$M_\phi = \int_0^1 \frac{1}{\gamma}(1-p)^{\frac{1}{\gamma}-1} q_x(p)dp \tag{7}$$

Wang's Distortion Function:

$$M_\phi = \int_0^1 e^{\left[-\alpha\Phi^{-1}(1-p)-\frac{\alpha^2}{2}\right]} q_x(p)dp \tag{8}$$

Where, p: Probability, $f(p)$: Distortion function of p, Φ: standard normal cumulative density function, and α, γ: Risk aversion coefficients. After comparing them, they conclude that Wang's and Dual Power functions are especially effective for different levels of risk aversion.

2. Data

2.1 Characterization of assumptions, portfolios and variables

To evaluate the spectral risk measure under normal and adverse conditions, estimates were made for March, April, May, and June of 2019 and 2020, excluding prior crises such as the 2008 crisis, as the EWMA methodology assigns higher weight to recent scenarios and lower weight to older ones. Time windows of 6 months, 1 year, 1.5 years, and 2 years were defined, with a daily horizon and confidence levels between 95% and 99.9%, with increments of 0.1%. The Spectral VaR, on the other hand, considered only confidence levels in the loss tail, between 50% and 99.9%, aiming for a broad estimation spectrum.

On the other hand, to compare the risk measurement methods, a simplified portfolio was built with a long position in global bonds in Colombian pesos (COP) payable in U.S. dollars (USD) and a short position in USD/COP forwards. A risk factor matrix was also constructed, including interest rates in U.S. dollars (USD) and Colombian pesos (COP), and the exchange rate series (TRM), separating market shocks. These variables, constructed with daily data from 02/01/2017 to 12/31/2020, were used to calibrate parameters and calculate returns and volatilities across the different windows applied.

2.2 Methodology and performing

2.2.1 Historical simulation

It was calculated using daily logarithmic return data, ordering the losses and selecting the percentile corresponding to the confidence level. This method does not assume a specific distribution for the data and is sensitive to the historical events that make up the analyzed series.

2.2.2 Parametric

This methodology adheres to the flow decomposition guidelines for financial assets as outlined by JPMorgan/Reuters (1996). It is assumed that returns follow a normal distribution. To calculate correlations between asset return series, which require the estimation of volatility and variance-covariance matrices, the exponential weighted moving average (EWMA) methodology described in the technical document by JPMorgan/Reuters (1996) was employed. This approach efficiently incorporates recent market shocks through the inclusion of the decay factor *Lambda*.

The decay factor *Lambda* λ is calibrated using daily logarithmic return data from January 2017 to October 2017 and will have a fixed value of 0.977264 throughout the study period, applying to the most recent data of the model. This method is computationally efficient but strongly depends on the assumptions of normality and volatility and correlation modeling.

2.2.3 Expected shortfall (ES)

Losses exceeding the Value at Risk (VaR) were averaged and adjusted according to the probability distribution selected in the parametric methodology for the corresponding confidence level. This approach satisfies the axiomatic properties of coherent risk measures and enhances coverage in extreme scenarios. However, it uniformly assigns equal weight to all losses exceeding the VaR threshold.

2.2.4 Spectral risk measures (SRM) with distorted functions

Spectral risk measures utilize distortion functions to assign weights to losses based on their severity and the degree of risk aversion, thereby attributing greater importance to extreme events. The resulting value at risk is derived from losses estimated through historical simulation and expected shortfall, increased by a weight calculated as the ratio between the $i + 1$ record and the total data n prioritizing higher losses, which are then transformed using the selected functions. The parameter associated with the investor's risk profile or risk aversion coefficient is defined with the Greek letter *Lambda* (λ) in Wang's distortion function and with the Greek letter *Beta* (β) in the Dual Power distortion function and takes values between 0.5 and 5.5 with increments of 0.5. This is done to evaluate how the VaR estimate changes with different risk profiles.

2.2.4.1 Wang's distortion function (Wang's D.F.): The Wang function used a transformation based on the Wang operator. Loss values were rescaled using an adjustment constant *Lambda* (λ) to modify the probabilities in the tail of the distribution, increasing the weight of more severe losses.

2.2.4.2 Dual power distortion function (Dual Power D.F.): The Dual Power function adjusted probabilities using a transformation based on a parameter *Beta* (β), which controls sensitivity to extreme losses. This approach allowed the modeling of more diversified risk profiles, adjusting the weights proportionally to the magnitude of the losses.

2.2.5 Backtesing the value at risk models: A backtesting test was performed by comparing the actual losses of the portfolio in 2019 and 2020 with those estimated by different models, using various time windows and confidence levels between 85% and 99.9%. The results were integrated into a heatmap to assess excesses within the allowed limit. The Kupiec Test was used to analyze the confidence and error probability of the model, evaluating the excesses as P&L losses exceeding those estimated.

3. Empirical results

3.1 General results

During the four months of 2019, the models performed similarly, with the parametric and expected shortfall models estimating higher losses and the historical simulation model estimating lower losses. In 2020, all models effectively incorporated the market shocks caused by the pandemic, particularly the spectral risk measure. Table 1 provides a breakdown of the results by model:

Table 1. VaR performance for 2019 y 2020 by Time Windows (MM COP)

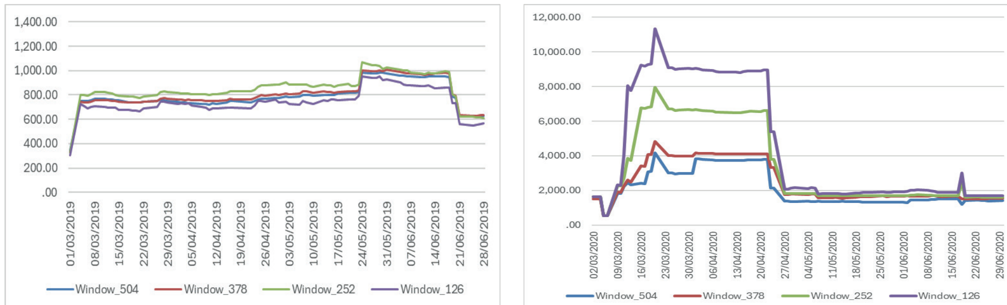
Method/ Window	2019_504	2019_378	2019_252	2019_126	2020_504	2020_378	2020_252	2020_126
Parametric	3.394,42	3.394,34	3.395,39	3.387,65	15.675,30	15.675,29	15.675,30	15.656,30
Expected Shortfall	3.888,87	3.888,78	3.889,98	3.881,11	17.958,64	17.958,63	17.958,64	17.936,87
Historical Simulation	2.322,94	2.278,78	2.309,01	1.854,46	2.086,36	2.451,49	3.392,67	4.426,60
SRM Wang HS	0.792,50	0.807,40	0.851,07	0.742,61	0.494,19	0.508,21	0.598,01	0.731,25
SRM Dual HS	0.788,25	0.804,11	0.848,36	0.736,93	0.469,95	0.487,11	0.560,87	0.709,12
SRM Wang ES	2.032,13	2.262,54	2.570,63	3.036,54	9.384,32	10.448,57	11.867,71	14.033,65
SRM Dual ES	2.028,43	2.258,08	2.564,82	3.027,67	9.367,20	10.427,95	11.840,86	13.992,66

Source: Author's own work.

3.2 Historical simulation

As expected, the indicator calculated with larger windows exhibits behavior like the parametric and expected shortfall methods, effectively incorporating the market shocks of 2020. However, its value at risk is significantly lower in monetary terms compared to other methods. Figure 1 illustrates this behavior:

Figure 1. VaR Estimation by Historical Simulation Across Different Time Windows for 2019 and 2020 with a 99% Confidence Level (mm cop)



Source: Author's own work.

3.3 Parametric and expected shortfall

An increase in VaR is noted in both methods during 2020, demonstrating the rapid response of this methodology in effectively integrating the market shocks caused by the pandemic. The figures (2) detail the behavior of CVaR during 2019 (left) and 2020 (right).

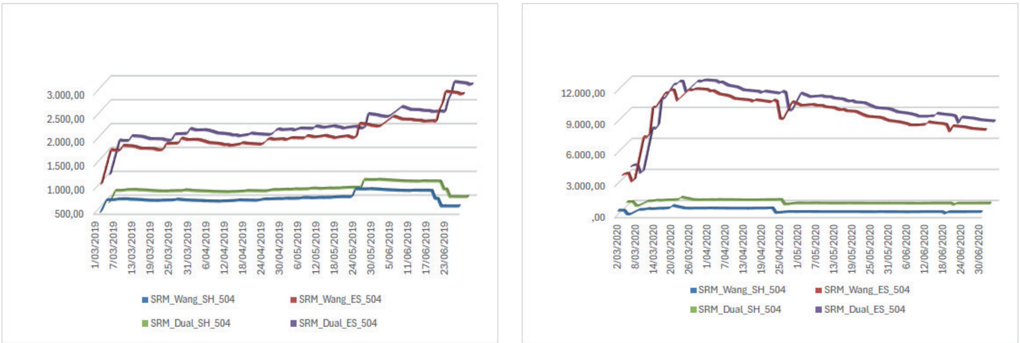
Figure 2. Expected Shortfall Estimation Across Different Time Windows for 2019 and 2020 with a 99% Confidence Level (MM COP)

Source: Author's own work.

3.4 Spectral Risk Measure including Distorted Functions

The Value-at-Risk (VaR) estimates, derived from distortion functions, demonstrate their effectiveness in modeling risk under both normal and uncertain market conditions, with a robust adjustment to market shocks. As shown in Figure 3 for the years 2019 and 2020, regardless of the distortion function applied, the values based on historical simulation are significantly lower than the average. In contrast, the spectral measure grounded in Expected Shortfall exhibits greater alignment with both the parametric model and the average results.

Figure 3. Comparative Analysis of Spectral Risk Measure with Distortion Functions for 2019 and 2020 Across Time Window 504 days (MM COP)



Source: Author's own work.

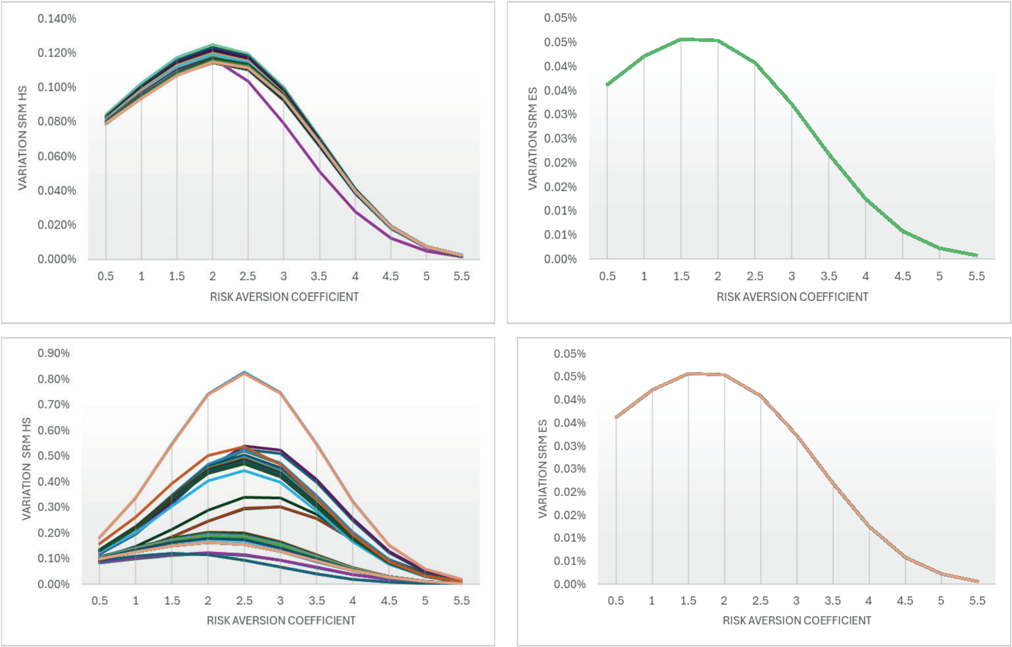
Similarly, estimates using the Dual Power function yield slightly lower losses than Wang's distortion function, both with historical simulation losses and expected shortfall. Additionally, larger time windows record lower values in the spectral risk measure. In both periods, under normal and uncertain conditions, the spectral measure produces lower monetary values than those estimated by previous methods, as it considers the entire loss tail to calculate the weighted average.

3.5 Spectral risk measure with Wang's distortion function

The loss series derived from historical simulation exhibits the lowest monetary values among the methodologies evaluated, attributed to the weighted average of the loss tail. In contrast, spectral risk measures employing the expected shortfall loss series demonstrate superior loss modeling and closer alignment with undistorted VaR estimates, establishing it as a promising approach for risk measurement in financial institutions. Moreover, a comparison between losses modeled by expected shortfall and those obtained through Wang's Distortion Function (Wang's D.F.) reveals that the latter achieves better alignment with the data, effectively mitigating underestimations and overestimations.

Additionally, analyzing the results based on the risk aversion coefficient reveals that higher values of the *Lambda* (λ) parameter result in greater losses, though the rate of increase diminishes after a certain point, consistent with the findings by Ho & Nguyen (2017). This behavior is observed for both the spectral measure based on historical simulation and that based on expected shortfall, under normal and uncertain conditions. Figure 4 illustrates this pattern, highlighting how losses increase with risk aversion, with the historical simulation results (left) and the expected shortfall results (right), considering a time window of 504 days.

Figure 4. Behavior of Estimating Losses by Wang's Distortion Function for 2019 and 2020 vs Agent Risk Profile



Source: Author's own work.

The results indicate that, for historical simulation, losses increase rapidly up to a risk aversion coefficient of *Lambda* (λ) = 2 under normal conditions (2019) and *Lambda* (λ) = 2.5 in adverse situations (2020). On the other hand, with expected shortfall, loss growth decelerates upon reaching *Lambda* (λ) = 1.5 in both years, suggesting that this model prioritizes more significant losses but moderates their impact when facing risks. Similarly, the behavior of the losses suggests that the range for the *Lambda* parameter should not exceed 5.5, indicating that assigning higher risks to the agent's risk aversion coefficient is not worthwhile.

Moreover, shorter windows show greater increases in losses per coefficient compared to longer windows, especially in the spectral measure based on expected shortfall. This highlights its sensitivity to short time horizons and underscores the importance of considering this characteristic when implementing such risk measures.

Table 2. VaR Performance with Wang's D.F. for 2019 and 2020 per Risk Aversion Coefficient (MM COP)

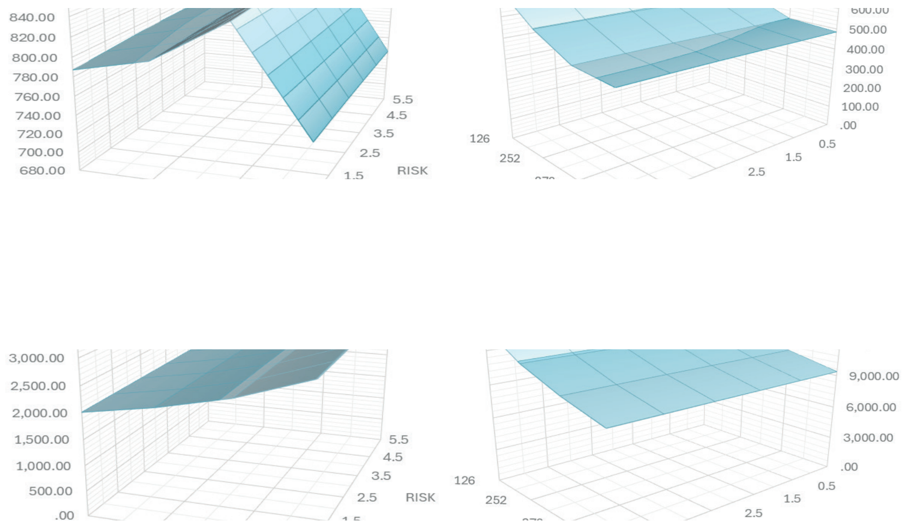
Spectral Risk Measure	Year	2019				2020			
	Risk	126	252	378	504	126	252	378	504
Historical Simulation	0,5	740,20	847,26	803,05	787,20	721,97	588,25	498,84	485,07
	1,5	741,26	848,66	804,52	788,84	725,14	590,99	501,15	487,17
	2,5	742,07	849,97	806,00	790,68	728,66	594,65	504,59	490,45
	3,5	742,48	850,77	806,99	791,97	730,64	597,12	507,18	493,08
	4,5	742,59	851,03	807,34	792,43	731,18	597,90	508,07	494,04
	5,5	742,61	851,08	807,41	792,50	731,26	598,01	508,21	494,20
Expected Shortfall	0,5	3.004,46	2.549,15	2.244,58	2.016,45	13.973,78	11.830,14	10.420,20	9.360,96
	1,5	3.009,68	2.552,18	2.246,78	2.018,23	13.998,04	11.844,20	10.430,41	9.369,18
	2,5	3.014,32	2.555,06	2.248,91	2.019,97	14.019,61	11.857,53	10.440,33	9.377,26
	3,5	3.016,67	2.556,70	2.250,21	2.021,06	14.030,57	11.865,15	10.446,34	9.382,33
	4,5	3.017,27	2.557,18	2.250,63	2.021,43	14.033,33	11.867,40	10.448,28	9.384,05
	5,5	3.017,34	2.557,25	2.250,69	2.021,49	14.033,65	11.867,71	10.448,57	9.384,32

Source: Author's own work.

On the other hand, organizing the losses relative to the risk aversion coefficient shows that the more risk-averse the investor's profile, the slightly higher the estimated loss. This observation holds true for both losses based on historical simulation and those using expected shortfall results.

Table 2 details the results by time window and risk aversion coefficient for the spectral risk measures calculated in 2019 and 2020, while Figure 5 illustrates the risk spectrum for each methodology.

Figure 5. Wang's SRM Spectrum – Average Loss in 2019 and 2020
per Risk Aversion Coefficient and Time Window (MM COP)

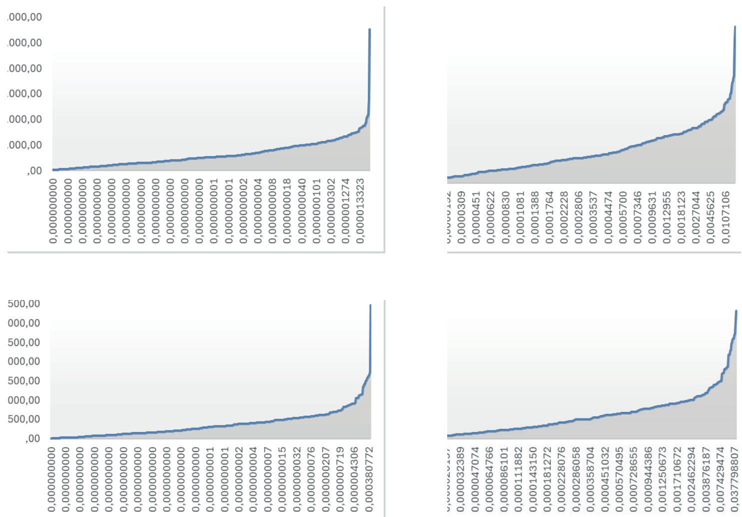


Source: Author's own work.

Additionally, it is confirmed that the weights applied to the loss series comply with the properties described by Wang (2002), showing a concave and increasing function whose sum equals one. For larger windows, the VaR is almost double that of smaller windows, reflecting the agent's assigned risk: high aversion concentrates weights on the largest losses, whereas neutral aversion distributes them more evenly. Moreover, in response to market shocks, the greatest weight is assigned to the most significant loss, and lower risk aversion generates steeper slopes in the weight distribution curve.

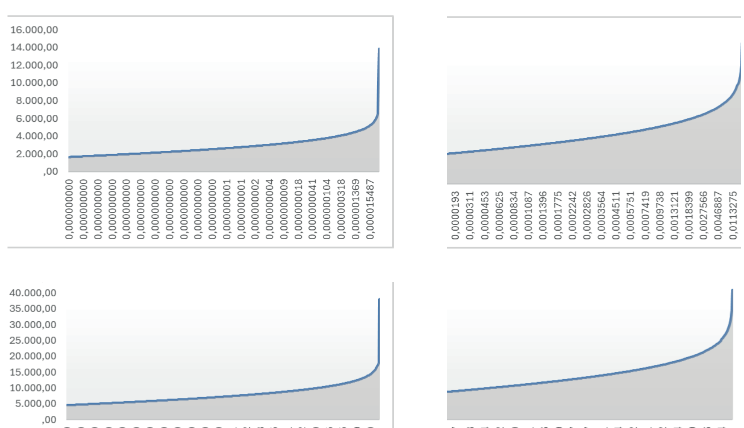
Figures 6 and 7 depict the weight assignment in 504-day time windows with risk aversion coefficients of *Lambda* (λ) = 2 and *Lambda* (λ) = 5.5 separated by methodology under normal conditions (2019) and adverse conditions (2020).

Figure 6. VaR vs. Weight Applied with Wang's Distortion Function; Historical Simulation Calculation for 2019 and 2020 (MM COP)



Source: Author's own work.

Figure 7. VaR vs. Weight Applied with Wang's Distortion Function; Expected Shortfall Calculations, for 2019 and 2020 (MM COP)



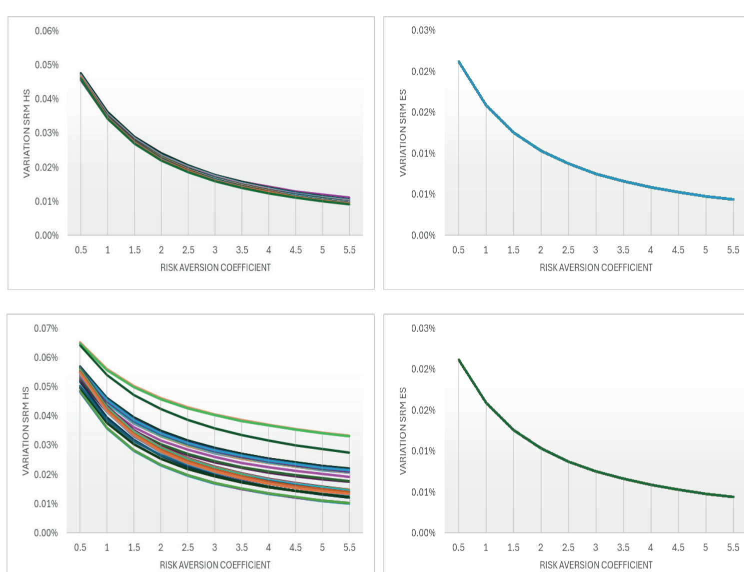
Source: Author's own work.

3.6 Spectral risk measure with dual power distortion function

The results exhibit behavior consistent with Wang's Distortion Function (Wang's D.F.), yielding lower estimated values compared to other methodologies due to the incorporation of the entire loss tail in the calculation of the weighted average. Although the estimates derived from the Dual Power Function (Dual Power D.F.) are marginally lower, the losses calculated using historical simulation consistently reflect the lowest monetary values among all approaches. These findings suggest that when implementing a spectral risk measure with distortion functions, particular care should be taken when integrating loss estimates derived from historical simulation.

A comparison of losses estimated through the Dual Power D.F., expected shortfall, and Wang's D.F. reveals that distortion functions offer a more precise reflection of potential portfolio losses under both normal and adverse market conditions, effectively mitigating the risk of underestimation or overestimation inherent in other models. This analysis underscores the advantages of spectral risk measures, emphasizing the critical importance of selecting appropriate distortion functions and their integration with loss estimates for improved accuracy in risk measurement techniques.

Figure 8. Behavior of Estimating Losses by Wang's Distortion Function for 2019 and 2020 vs Agent Risk Profile



Source: Author's own work.

Figure 8 illustrates the behavior of losses as the risk aversion coefficient increases. The results indicate that losses rise with the *Beta* risk aversion coefficient, though the growth rate diminishes as the coefficient increases—a common pattern across all cases. This behavior is observed for both the spectral measure of historical simulation and expected shortfall under normal and uncertain conditions.

Comparing these results with those obtained using Wang's D.F. it should be noted that losses under this function exhibit rapid growth up to a specific threshold, after which they begin to decline. In contrast, the Dual Power D.F. demonstrates a more moderate growth trajectory following the initial risk aversion coefficient. These findings confirm the results of Ho & Nguyen (2017) and suggest that the spectral measure with Wang's D.F. is suitable for agents with high risk aversion, while the Dual Power D.F. might be more appropriate for agents with prior experience in market risks.

Table 3. VaR performance with Dual Power D.F. for 2019 and 2020 per Risk Aversion Coefficient (MM COP)

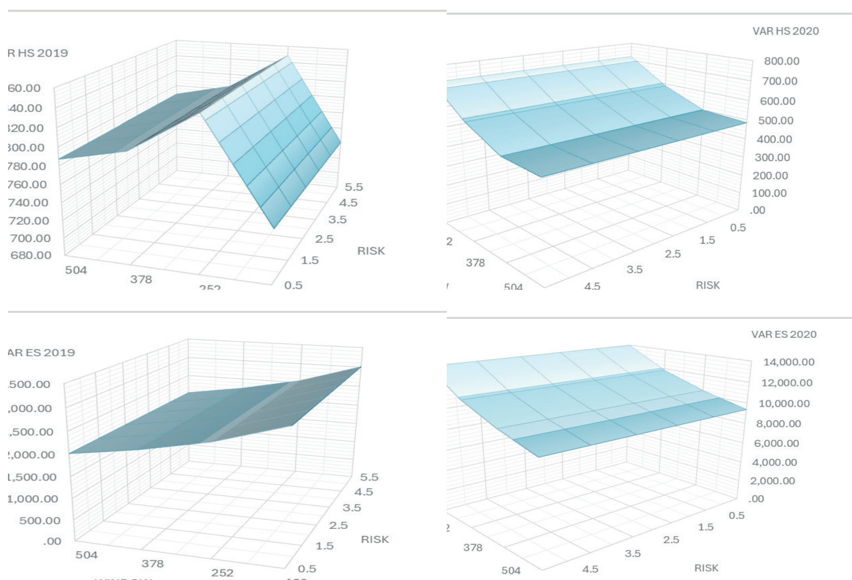
Spectral Risk Measure	Year	2019				2020			
	Risk	126	252	378	504	126	252	378	504
Historical Simulation	0,5	740,01	847,02	802,79	787,20	721,43	587,83	498,50	484,98
	1,5	740,44	847,50	803,28	787,60	722,14	588,35	498,92	485,30
	2,5	740,71	847,82	803,60	787,88	722,70	588,76	499,23	485,56
	3,5	740,89	848,04	803,84	788,10	723,17	589,10	499,49	485,77
	4,5	741,03	848,22	804,02	788,28	723,58	589,39	499,72	485,96
	5,5	741,14	848,36	804,17	788,36	723,94	589,66	499,92	486,04
Expected Shortfall	0,5	3.003,51	2.548,62	2.244,19	2.016,15	13.969,34	11.827,65	10.418,42	9.359,53
	1,5	3.005,22	2.549,59	2.244,90	2.016,72	13.977,30	11.832,18	10.421,70	9.362,19
	2,5	3.006,39	2.550,25	2.245,38	2.017,11	13.982,72	11.835,25	10.423,92	9.363,98
	3,5	3.007,26	2.550,75	2.245,73	2.017,39	13.986,78	11.837,54	10.425,57	9.365,30
	4,5	3.007,95	2.551,14	2.246,02	2.017,62	13.990,00	11.839,36	10.426,88	9.366,35
	5,5	3.008,52	2.551,46	2.246,25	2.017,80	13.992,66	11.840,86	10.427,95	9.367,20

Source: Author's own work.

On the other hand, when evaluating losses by time window, it is observed that shorter time windows show greater increases in losses compared to longer windows, especially in the spectral measure based on expected shortfall. Conversely, for the spectral measure based on historical simulation, increases in losses due to higher risk aversion coefficients occur in longer time windows under normal conditions (2019), while under uncertainty conditions (2020), they appear in shorter time windows. Additionally, when organizing losses relative to the risk aversion coefficient, as done with Wang's D.F. results, it is evident that the more risk-averse the investor's profile, the slightly higher the estimated loss.

Table 3 details the results by time window and the *Beta* parameter representing the risk aversion coefficient for the spectral risk measures calculated in 2019 and 2020, while the risk spectrum for the Dual Power Function is illustrated in Figure 9.

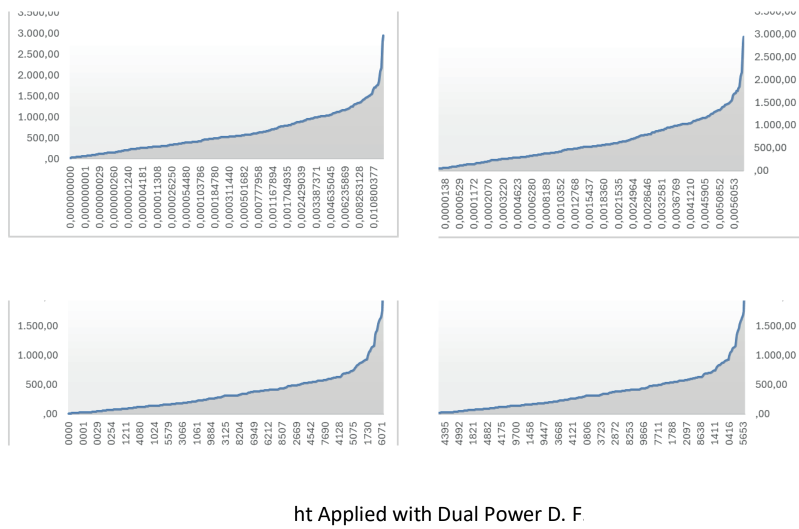
Figure 9. SRM Dual Power Function Spectrum - Average Loss in 2019 and 2020 per risk Aversion Coefficient and Time Window (MM COP)



Source: Author's own work.

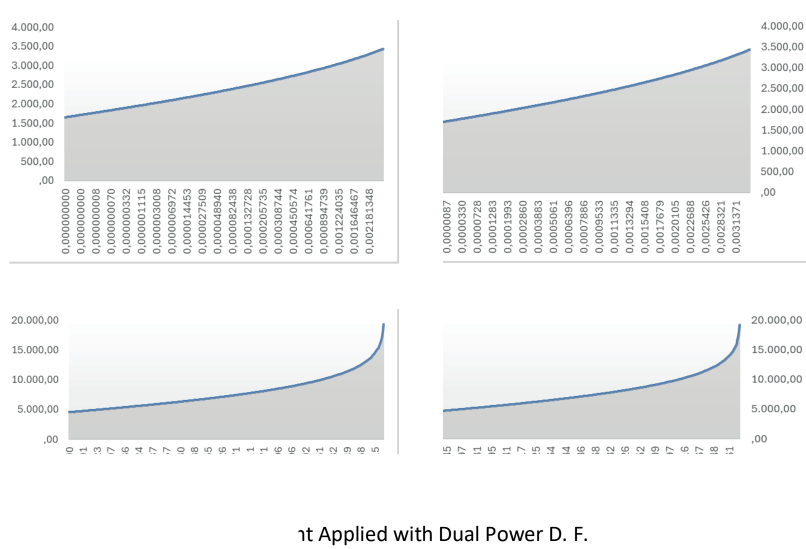
Finally, evaluating the weights applied to the loss series, these satisfy the properties of a concave and increasing function, as described by Balbás et al. (2008), ensuring that the sum of the weights equals one, which validates their coherence as a distorted risk measure. Analyzing the behavior of VaR with the Dual Power Function, it is observed that weights are distributed homogeneously across losses. Under high risk aversion, 50% of the weights are assigned to the largest losses, whereas, with neutral risk aversion, these weights are distributed across a broader range of losses.

Figure 10. VaR vs. Weight Applied with Dual Power Distortion Function; Historical Simulation Calculation for 2019 and 2020 (MM COP)



Sou

Figure 11. VaR vs. Weight Applied with Dual Power Distortion Function; Expected Shortfall Calculations, for 2019 and 2020 (MM COP)



Sou:

Figures 10 and 11 depict the weight distribution for 504-day time windows with risk aversion coefficients *Beta* (β) = 2, and *Beta* (β) = 5,5 separated by methodology under normal conditions (2019) and adverse conditions (2020).

This behavior reflects how an agent assigns greater weight to the most significant losses when highly risk-averse, while distributing weights more evenly in a neutral risk aversion context. Although both distortion functions assign greater weights to the largest losses, Wang's D.F. is more rigid in setting weights, making it more suitable for highly risk-averse profiles. In contrast, the Dual Power D.F. is more flexible and appropriate for less risk-averse profiles due to its more balanced weight distribution. Furthermore, it is inferred that, in scenarios of uncertainty with strong market shocks, increases in the risk aversion coefficient result in a steeper curve in the Dual Power D.F. compared to Wang's D.F.

3.7 Backtesting results

The results indicate that, overall, all methodologies are suitable for estimating VaR, except for the historical simulation method under adverse conditions, as its estimates are the smallest and result in breaches when compared to actual losses. On the other hand, spectral risk measures that use expected shortfall results adequately estimate losses, matching or even exceeding the estimates of the methodology without distortion functions, both under normal and adverse conditions.

Additionally, it is observed that, for spectral risk measures, the confidence level spectrum is at the extremes, accepting the null hypothesis (H_0) below 88% or above 95% confidence, in both the 252-day and 504-day windows. Table 4 provides detailed results of the backtesting performed:

4. Conclusions and extensions

The framework presented is crucial for addressing the inconsistencies observed in traditional risk measures such as VaR and CVaR. By incorporating distortion functions, it offers more precise and consistent tools, which are fundamental for evaluating financial and actuarial risks in a comprehensive and context-adjusted manner.

The implementation of distortion functions for estimating risk in a portfolio of Forwards and Global Bonds yields optimal results compared to methods like Historical Simulation, Parametric, or Expected Shortfall, all of them under

Table 4. Comparison of excesses recorded by methodology vs. permitted excesses; window 252 and 504 days

252	2019								2020							
Level	Exc.	Excesses Recorded							Exc.	Excesses Recorded						
Conf.	Max.	HS	VaRP	ES	Wang HS	Wang ES	Dual HS	Dual ES	Max.	HS	VaRP	ES	Wang HS	Wang ES	Dual HS	Dual ES
99.9%	2	0	0	0	14	0	14	0	2	2	1	1	22	1	22	1
99.5%	3	0	0	0	14	0	14	0	3	2	1	1	22	1	22	1
99.0%	3	0	0	0	14	0	14	0	3	3	1	1	22	1	22	1
98.0%	5	1	0	0	14	0	14	0	5	5	1	1	22	1	22	1
97.0%	6	1	0	0	14	0	14	0	6	10	1	1	22	1	22	1
96.0%	7	4	0	0	14	0	14	0	7	11	1	1	22	1	22	1
95.0%	8	5	1	0	14	0	14	0	8	13	1	1	22	1	22	1
94.0%	9	5	1	0	14	0	14	0	9	14	1	1	22	1	22	1
93.0%	10	5	1	0	14	0	14	0	9	17	1	1	22	1	22	1
92.0%	10	5	1	0	14	0	14	0	10	17	1	1	22	1	22	1
91.0%	11	6	3	0	14	0	14	0	11	17	1	1	22	1	22	1
90.0%	12	7	4	0	14	0	14	0	12	17	1	1	22	1	22	1
89.0%	13	7	4	0	14	0	14	0	13	17	1	1	22	1	22	1
88.0%	14	7	5	0	14	0	14	0	14	19	1	1	22	1	22	1
87.0%	15	7	5	1	14	0	14	0	14	20	1	1	22	1	22	1
86.0%	15	7	5	1	14	0	14	0	15	21	1	1	22	1	22	1
85.0%	15	7	5	1	14	0	14	0	15	21	1	1	22	1	22	1
504	2019								2020							
Level	Exc.	Excesses Recorded							Exc.	Excesses Recorded						
Conf.	Max.	HS	VaRP	ES	Wang HS	Wang ES	Dual HS	Dual ES	Max.	HS	VaRP	ES	Wang HS	Wang ES	Dual HS	Dual ES
99.9%	2	0	0	0	16	1	16	1	2	2	1	1	25	1	25	1
99.5%	3	0	0	0	16	1	16	1	3	3	1	1	25	1	25	1
99.0%	3	0	0	0	16	1	16	1	3	5	1	1	25	1	25	1
98.0%	5	1	0	0	16	1	16	1	5	10	1	1	25	1	25	1
97.0%	6	3	0	0	16	1	16	1	6	13	1	1	25	1	25	1
96.0%	7	5	0	0	16	1	16	1	7	16	1	1	25	1	25	1
95.0%	8	5	1	0	16	1	16	1	8	17	1	1	25	1	25	1
94.0%	9	5	1	0	16	1	16	1	9	17	1	1	25	1	25	1
93.0%	10	6	1	0	16	1	16	1	9	17	1	1	25	1	25	1
92.0%	10	7	1	0	16	1	16	1	10	18	1	1	25	1	25	1
91.0%	11	7	3	0	16	1	16	1	11	18	1	1	25	1	25	1
90.0%	12	7	4	0	16	1	16	1	12	19	1	1	25	1	25	1
89.0%	13	7	4	0	16	1	16	1	13	20	1	1	25	1	25	1
88.0%	14	7	5	0	16	1	16	1	14	20	1	1	25	1	25	1
87.0%	15	8	5	1	16	1	16	1	14	21	1	1	25	1	25	1
86.0%	15	11	5	1	16	1	16	1	15	21	1	1	25	1	25	1
85.0%	15	12	5	1	16	1	16	1	15	22	1	1	25	1	25	1

Source: Author's own work.

normal and uncertain conditions. This is due to its ability to provide slightly lower values than the Parametric and Expected Shortfall methods, as it evaluates the entire loss tail to obtain a weighted average. Among the evaluated models, Expected Shortfall demonstrates superior performance, while Historical Simulation provides inadequate results due to its underestimation of losses and excesses during backtesting.

Under normal conditions, spectral measures utilizing distortion functions—regardless of the specific function—accurately model losses and adapt to specific investor risk profiles. In uncertainty conditions, the Expected Shortfall model combined with the Wang Distortion Function emerges as the most appropriate, as it corrects the confidence spectra, appropriately models losses without underestimations or overestimations, and accurately adjusts to diverse investor risk profiles.

The spectral risk measure proves to be a strong option for financial institutions, as it accurately models losses and aligns with various risk profiles. Consequently, the Wang Distortion Function is suited for risk-averse or inexperienced agents, while the Dual Power Distortion Function is more suitable for investors with neutral or experienced risk profiles. This distinction arises because the Wang Distortion Function assigns a greater weight to larger losses (90% to the most significant loss), while the Dual Power Distortion Function distributes the weights more evenly, allocating half of the total weight to the top 20% of losses.

Future research should focus on calibrating key parameters such as lambda and Beta of the risk profile, linking them to systematic risk, and exploring new methodologies for assessing volatility and capital requirements. Additionally, further studies should examine the calibration of parameters developed in this paper and consider different probability distribution functions to evaluate them with consistent risk measures.

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