# Chaos Theory and the Science of Fractals in Finance

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## Introduction

The evolution of the financial environment since 1970 has come with a very high price. Without clear warnings, at least eight major crises have struck the financial sector. The subprime crisis, which originated in the United States in 2007, can be considered one of the worst financial disasters. For the first time, problems that originated in one country had global effects leading the world' economy to a serious recession. However, it is in these critical times that beliefs are revaluated and new paradigms emerge to guarantee that next time crises come at a discounted value.

Indeed, one of the mayor lessons of the subprime crisis has been that current financial models are not based on adequate assumptions. Evidently, Neoclassical theory, today's mainstream financial paradigm, has become obsolete for explaining the complexity of financial markets. It oversimplifies reality, and thus, can only address problems under ideal or normal conditions. For this reason, it is necessary to look for alternative theories that allow the description of the real market dynamics, and provide accurate tools to measure the apparent disorder of today's capital markets.

Consequently, this article proposes the application of Chaos Theory and the Science of Fractals to finance. Within this new framework, it would be possible to find a more general, but coherent perspective to study financial phenomena. Mainly, it focuses on the fractal structure of capital markets to be able to develop new analytical and mathematical tools. This would allow financial analysts to understand better financial behavior.

## I. The Development of Chaos Theory and the Science of Fractals in Science

Chaos Theory was developed in physics with the study of complex systems and fractal structures in nature. However, before this theory consolidated as a main paradigm in science, many preconceived ideas had to be changed. In particular, Newton's ideas of the universe and nature, which were deeply rooted in the discourse and method of scientists for more than a hundred years. The most significant changes came in the nineteenth century with the science of heat and quantum mechanics. Nevertheless, even Darwin's evolution theory and Einstein's relativism help to set ground for the emergence of a new synthesis in science. The result of this evolution was Chaos Theory. This new paradigm proposes a new language and tools to address our complex world.

## A. From Newton's Time to the Quantum World

From the seventeenth century to the nineteenth century, Newton's intellectual contributions influenced the scientific discourse and method. In 1687, Newton published his book *Philosophiæ Naturalis Principia Mathematica*, which included the laws of motion[1] and the universal force of gravitation[2]. With this book he explained the mechanics of world with laws that were "independent in time", in the sense that past and future play the same role, and universally deterministic[3]. Newton's idea of the universe is, therefore, represented as a clockwork machine where everything can be predicted in advance, or looked upon in retrospect.

This mechanical interpretation of the universe is observed in both Newton's discourse and approach to science. As the inventor of the calculus, Newton was able to simplify natural phenomena into linear equations and other predictable formulas. Even more, his theorems followed a mathematical logical system similar to the method of Euclidean geometry[4], where few elegant simple premises, combined with a deductive logic, revealed the "truth" of the universe. Thus, Newtonian laws are characterized by their mathematical formalism and methodological reductionism.

1 The law of motion comprises three main postulates: 1) every object is at a state of rest unless some external force acts upon it changing its uniform motion; 2) the object will move in the same direction and proportional to the force applied [Force = Mass * Acceleration]; and 3) every action has an equal and opposite reaction. Thus if the body 1 exerts on body 2 a Force A, then body 2 exerts on body 1 a Force B, resulting in $F(A) = F(B)$.

2 The law of gravitation is a special case of the second postulate. This law states that there is an attraction between all bodies with mass, which is directly proportional to the quantity of matter that they contain, and inversely proportional to the square of the distance between them.

3 In Newtonian physics, the world is deterministic, meaning that every state in the world is the result of a preceding occurrence.

4 Euclidean Geometry is a mathematical system named after the Greek mathematician Euclid of Alexandria. In his famous book *Elements*, Euclid organized and summarized the geometry of ancient Greece based on lines, planes and spheres. In his discussion about geometry, thus, he describes the smoothness and symmetry of nature in idealized or Platonic terms. However, Euclid's great contribution was his use of a deductive system for the presentation of mathematics. As so, he explained a series of theorems all derived from a small number of axioms. Although Euclid's system no longer satisfies modern requirements of logical rigor, its importance in influencing the direction and method of the development of mathematics is undeniable. Indeed, Euclidean Geometry remained unchallenged until the middle or late nineteenth century.

For more than a century, Newton's propositions were considered the ultimate science and its mathematics the ultimate expression of reality. However, discoveries in the cosmic and natural world showed that the universe was in fact govern by more complex processes than those observed by Newtonian mechanics. This led scientists to question existing ideas, and to embrace a more organic view of nature, less orderly and less predictable.

The first rupture was made with the Second Law of Thermodynamics[5]. In 1824, Sadi Carnot published the first statement of what later developed into this law. Carnot argued that heat could not spontaneously flow from cold objects to hot ones. More specifically, heat could flow from a higher to a lower temperature, but never in reverse direction, except with the action of an external force. As opposed to Newtonian mechanics, this demonstrated that some processes are simply irreversible in time. Indeed, "if there were no second law, the universe would be like a giant clock that never run down" (Lightman, 2000, 63).

The second rupture came from the development of quantum mechanics[6]. The main contribution of this field is that it changed the deterministic Newtonian perspective and gave science a natural uncertainty that could only be described by states of probability. For instance, the German physicist Heisenberg proposed the *uncertainty principle* as he discovered that is not possible to determine an exact position and moment of an object simultaneously. More generally, this principle argues that it is unlikely to know with precision the values of all the properties of a system at the same time. In this way, the properties that are not possible to be described can only be inferred by probabilities. At last, even in exact sciences such as mathematics there is not total certainty[7].

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5 Thermodynamics is a branch of physics that deals with the various phenomena of energy and related properties of matter, especially the laws of transformation of heat from one form to another. The basis of this science is experimental observation.

6 The term *quantum* comes from Max Planck's suggestion in 1900 that radiant energy was emitted from a heated body in discontinuous portions that he termed *quanta*. Afterwards, the term quantum mechanics was generalized to define a branch of physics that studies the physical reality at the atomic level of matter.

7 Kurt Gödel will complement this idea with his *incompleteness theorem*. In 1931, Gödel demonstrated that there were limits in mathematics, because there are problems that do not have an established solution. In other words, for a given group of postulates there is always going to be one of them whose validity can neither be proven nor disproven.

Moreover, Niels Bohr, Danish physicist, developed the *concept of complementary*, which states that subatomic particles have simultaneous wave and particle properties. However, because it is not possible to observe both states at the same time, it is the observer who determines the properties of an object. This implies that measuring instruments affect the behavior of atomic objects and the conditions under which certain phenomena appear. In 1927 Niels Bohr wrote: "Anyone who is not shocked by quantum theory does not understand it" (Rosenblum and Kuttner, 2006, 52).

Evidently, quantum mechanics contradicts the Newtonian clockwork machine notion. With this "new science"[8], "the universe begins to look more like a great thought than a great machine" (Rosenblum and Kuttner, 2006, 51). Nevertheless, accepting quantum theory means confronting a big enigma as it postulates ultimate randomness in nature.

Chaos Theory is the final rupture with Newtonian mechanics. This theory studies systems that appear to follow a random behavior, but indeed are part of a deterministic process. Its random nature is given by their characteristic sensitivity to initial conditions that drives the system to unpredictable dynamics. However, in a chaotic system, this non-linear behavior is always limited by a higher deterministic structure. For this reason, there is always an underlying order in the apparent random dynamics. In Sardar and Abrams (2005) words: "In Chaos there is order, and in order there lies chaos" (Sardar and Abrams, 2005, 18).

## B. Chaos Theory

Kellert (1993) defines chaos theory as "the qualitative study of unstable aperiodic[9] behavior in deterministic nonlinear[10] dynamical systems" (in Mc Bride, 2005, 235). Chaotic systems are

8 The "new" science is a general term for all the theories and ideas generated in different academic disciplines that do not correspond to the "classical" scientific explanation. Quantum mechanics is part of this approach to science, with other disciplines and theories, which leave behind the idea of the Newtonian world to model our complex reality

9 An aperiodic system is the one that does not experience a completely regular repetition of values. However the variables associated with the system are limited by a fixed and definable space.

10 The meaning of the concept of nonlinearity is that the whole is greater that or less than the sum of its parts. By contrast in a linear world, the effects are always proportional to their causes.

said to be mathematically deterministic because if the initial measurements were certain it would be possible to derive the end point of their trajectories. Nevertheless, chaotic systems have two important characteristics: 1) they are highly sensitive to changes in the initial conditions; and 2) they involve nonlinear feedback forces[11] that can produce unexpected results.

The investigation on this field can be trace to the early 1900's when the physicist Henri Poincare observed that a very small difference in the starting positions and velocities of the planets could actually grow to an enormous effect in the later motion. This was a proof that uncertainty would remain huge in certain systems, even though initial measurements could be specified with high precision. Poincare's discovery was neglected for many decades as it clearly contradicted the mechanistic Newtonian perspective.

However, in the early 1960's a small part of the scientific community became simple dissatisfied with existing paradigms. They often ignored important aspects to uphold linear equations and a reductionist method. It became then apparent that Newtonian mechanics had serious limitations in explaining the complexity of the universe. Therefore, scientists started looking for new explanations and new approaches that were more coherent with the organic nature of the world. The advent of computers facilitated this task, and thus the scientific community progressively turned to the study of non-linear dynamics, patterns and other complex behavior that were excluded in classical science.

A significant progress in the emergence of the "new science" came with Edward Lorenz's proposition of the *Butterfly Effect* in 1963. In his meteorological investigations, Lorenz found that variations in the decimals of initial measurements of the weather predicted a completely different motion. To illustrate better his theory, Lorenz gave the example of how a butterfly that flags its wings in Brazil can cause a tornado in Texas, indicating that a small wind could change the weather in few weeks. This is known as the *Butterfly Effect*, the "signature" of chaos theory, and it makes the point that some systems are highly sensitive to insignificant changes in the starting conditions.

Gradually, physicists discovered that most natural systems are characterized by local randomness as well as global determinism. These two states can coexist, because randomness induces the local innovation and varie-

11 Feedback is present in a system when the inputs affect the outcome, and at the same time, the outcome will influence the inputs.

ty, but determinism gives the global structure. Therefore, a random system behaves always within certain bounds.

Overall, the objective of Chaos Theory is to study changing environments, full of nonlinear dynamics, discontinuities, feedback systems and intelligible, but not predictable, patterns. This influenced deeply the discourse of science, allowing it to move upwards to a better understanding of the physical world. Most important, it triggered a significant change in its methodology. According to Mirowski (2004):

> "The breakthrough came when physicists stopped looking for deterministic invariants and began looking at geometric patterns in phase space. What they found was a wholly different kind of order amid the chaos, the phenomenon of self-similarity at different geometric scales. This suggested that many phase-space portraits of dynamical systems exhibited fractal geometry; and this in turn was taken as an indication that a wholly different approach must be taken to describing the evolution of mechanical systems" (Miroswki, 2004, 243)

Chaos Theory, therefore, suggests the Science of Fractals as the framework where new tools can be found and new ways of solving problems can be explored. In this way, as Euclidean Geometry served as a descriptive language for the Newtonian mechanics of motion, fractal geometry is being used for the patterns produced by chaos.

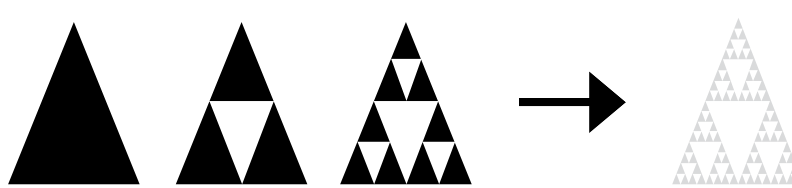

**Figure 1.1** *The Sierpinski triangle* is a fractal generated by connecting the midpoints of a usual triangle to form four separate triangles (the one in the center is later cut). This process is repeated infinite times until the final figure is observed.

## C. The Science of Fractals

Benoit Mandelbrot developed the field of fractal geometry between 1970 and 1980 with books such as *Fractals: Forms, Chance and Dimensions* (1977) and *Fractal Geometry of Nature* (1982). A fractal[12] is a shape made of parts similar to the whole in some way, thus they look (approxi-

12 The term "fractal" was coined by Benoit Mandelbrot and was derived from the Latin word *fractus*, which means "broken" or "fractured".

mately) the same whatever scale they are observed. When fractals scale up or down by the same amount, they are said to be self-similar. In the contrary, if they scale more in one direction than another, they are called self–affine. And in their most complex form, they are named multifractals, which scale in many different dimensions and ways. This diversity of fractals allows them to be found in nature, the human body, and art, between others.

The concept of fractals is inextricably connected to the notion of fractal dimension[13]. In Euclidean mathematics a point had one dimension, a line two and a cube three. With Einstein, and his Relativity theory, the physics world added time as the fourth dimension. However in fractal science, the dimension depends on the point of view of the observer. "The same object can have more than one dimension, depending on how you measure it and what you can do with it. A dimension needs not to be a whole number; it can be fractional. Now and ancient concept, dimension, becomes thoroughly modern" (Mandelbrot, 2004, 129). The fractal dimension is important because it recognizes that a process can be somewhere between deterministic or random.

In spite of this, fractal geometry is in fact a simplifying and logical tool. In mathematics, fractal functions work like chaotic systems where random changes in the starting values can modify the value of the function in unpredictable ways within the system boundaries. The famous Mandelbrot set demonstrates this connection between fractals and chaos theory, as from a very simple mathematical feedback equation, highly complex results are produced (see Figure 1.2).

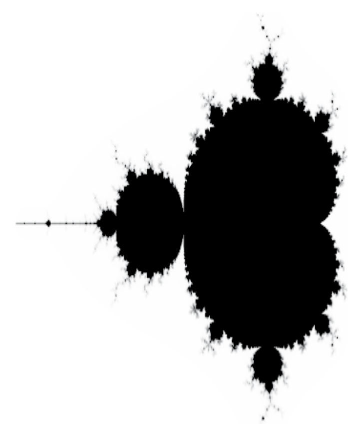

**Figure 1.2:** *Mandelbrot **Set.*** The Mandelbrot set embodies the principle of the transition from order to chaos. This fractal figure is the result of the process zn+1= zn^2 + c with different values of the constant c.

The key to understand fractals is then to discern those fundamental properties that do not change from one object

13 The fractal dimension gives a qualitative measure of the degree of roughness, brokenness or irregularity of a fractal.

under study to another[14]. Accordingly, "fractal geometry is about spotting repeating patterns, analyze them, quantify them and manipulate them, it is a tool of both analysis and synthesis" (Mandelbrot, 2004, 126). Given these special features, Fractal Geometry has extended to areas such as hydrology, meteorology and geology, and even to economics and finance.

## II. Chaos Theory and the Science of Fractals in Finance

Historically, it has been observed that the ideas that predominate in science influence the economic paradigms of the same period of time. Thus, the legacy of Newtonian mechanics is observed in the economic and financial ideas of the nineteenth century, in particular Neoclassical Theory. However, even though scientific ideas have evolved, mainstream financial theory is still connected to the classical logic of the world. For this reason, financial models are based on rigid assumptions, mathematical formalism and methodological reductionism. It is necessary, therefore, to explore a more coherent approach to finance that includes the perspective of contemporary science, Chaos Theory and the Science of Fractals.

### A. Neoclassical Theory

The theoretical background of Neoclassical Theory is found in the classical postulates of Adam Smith (the invisible hand) and the school of Utilitarianism[15]. However, it was properly developed in 1870 by Carl Menger, William Stanley Jevons and Leon Walras. Later, the economist Alfred Marshall will codify neoclassical economic thought in his book *"Principles of Economics"* published in 1890.

Although Neoclassical Theory has become an "umbrella" for other economic postulates, it shares the following core assumptions:

- People have rational and homogeneous preferences, thus they will always choose the outcome with the highest value;
- Individuals act as rational agents by maximizing their utility at any given point in time;

14 In Mandelbrot's words, "the key is to spot the regularity inside the irregular, the pattern in the formless" (Mandelbrot, 2004, 125).

15 Utilitarianism is an economic theory that explains the value of a product in terms of the different utility functions of consumers. Neoclassicals were influenced by their idea that utility can be measured, and because of this, market participants can act as rational individuals.

- Agents will act independently based on the relevant available information.

Neoclassical models further assume perfect competition in the market, no information asymmetry and no transaction costs.

Neoclassical Theory postulates that perfectly informed rational individuals, who just differ with respect to their utility functions, act as self-interested agents trying to optimize their resources. The market, as a result, will try to reconcile these conflicting desires until it reaches a state of equilibrium. In this point, known as *Pareto optimum*, any change will imply the destabilization of the system. For this reason, this school of thought concludes that markets will allocate scarce resources efficiently via the interaction of demand and supply.

Based on this idea of market behavior, Neoclassical theory built a structure to understand the functioning of all the markets in the economy, including financial markets. Consequently, from this neoclassical perspective, financial theories of competitive equilibrium, such as the Efficient Market Hypothesis and Random Walk Theory, will be developed.

## 1. The Efficient Market Hypothesis and the Random Walk Theory

The research on financial markets can be traced to the pioneer work of Louis Bachelier. In his Ph.D dissertation titled "*The Theory of Speculation"* (1900), Bachelier offered the first statistical method for analyzing the random behavior of stocks, bonds, and options. The main insight of his investigation was that in a fair market[16], price changes in either direction or a given amount have the same likelihood of occurrence. Consequently, the mathematical expectation of any speculator is zero[17]. "Under these conditions, one may allow that the probability of a spread greater than the true price is independent of the arithmetic amount of the price and that the curve of probabilities is symmetrical about the true price" (Bachelier 1900 p 28).

In spite of Bachelier's remarkable contribution, the interest in the analysis of the market from this point of

16 As it is observed, Bachelier assumes a fair market where influences that determine fluctuations are reflected in prices.

17 This idea implies that if an initial investment equals to \$100, it would be equally probably that at the end of the period the value moves to \$100 + k or \$ 100 – k. In other words, the probability of a gain is equal to the probability of loss, and hence the expected value in any future time remains \$100 and the expected gain equals to 0.

view developed very slowly. Just with the unprecedented crash of the stock market of 1929, academics began paying attention to his propositions. Afterwards, the study of random markets was mainly conducted to prove that the investment world was highly competitive, and hence prices reflected the best information about the future. Nonetheless, researchers did not go beyond this basic intuition and model the economics involved. To a large extent, the empirical work in this area preceded the development of a proper economic discourse.

In 1970 Eugene Fama developed the theoretical framework for random markets known as the *Efficient Market Hypothesis* (EMH). This theory states that a market is efficient in the determination of a "fair" price when all available information[18] is instantly processed as soon as it reaches the market, and is immediately reflected in the value of traded assets. Fama stated that for a market to be efficient, it must satisfy the following conditions: "1) there are no transaction costs in trading securities; 2) all available information is costessly available to all market participants; 3) all agree in the implications of current information for the current price and distributions of each security" (Fama, 1970, 387)[19].

In such frictionless world, investors earn a competitive expected return in the market, as all the cost and benefits associated with a value are already incorporated in the price. According to Fama, the competition between sophisticated investors allows the stock market to consistently price stocks in accordance with the best expectations of the future economic prospects (in Glen, 2005, 92-93). Thus, if the price deviates from its fundamental value,

18 Fama distinguished three forms of market efficiency. The *weak form*, which stipulates that current asset prices already incorporate past price and volume information. This means that investors cannot use historical data to predict future prices. For this reason technical analysis are not useful to produce excess returns and thus some fundamental analysis is required. The *semi-strong form* that argues that all the publicly available information is instantly reflected in a new price of the assets. In this case, not even fundamental analysis will be helpful to produce excess returns. And finally the *strong form*, which assumes that prices reflect all available information. Hence prices not only take into account historical and public information, but also private information. However, generally the term "all available information" represents the idea that prices reflect all ***publicly*** available information.

19 In this point it is important to highlight that Fama does acknowledge that frictionless markets are not met in practice. Therefore, these conditions are sufficient but not necessary. As long as in general prices fully reflect information, there is a "sufficient number" of investors that have immediate access to information, and disagreement among investors do not imply an inefficient market, then the EMH can still reflect rational market behavior.

market participants will correct it. At the end, prices will be in a competitive equilibrium with respect to information.

The Random Walk Hypothesis is an extension of the EMH. It states that random information is the only cause for changes in prices. Therefore, in the absence of new information there is no reason to expect any movement in the price, and the best possible forecast of the price of tomorrow will be, then, the price of today. As a result, the probability that changes occur can be as well determined by a chance game such as tossing a coin to obtain head or tails. This implies that the next move of the speculative price is independent of all past moves or events[20] or, as in mathematics, they form a sequence of identically and independently distributed (i.i.d) random variables.

As mentioned before, this was already formulated in Bachelier's attempt to mathematically explain a random walk in security prices. Fifty years later, Osborne expanded Bachelier's ideas and developed in 1959 the Bachelier-Osborne model. This model takes the idea of independence, and further assumes that "transactions are fairly uniformly spread across time, and that the distribution of price changes from transaction to transaction has finite variance" (Fama, 1965, 41). If the number of transactions is large, the distribution of i.i.d random variables will conform to the normal or Gaussian[21] distribution due to the Central Limit Theorem[22]. The normal distribution has the following desirable proper-

20 In statistics, independence means that "the probability distribution for the price changes during time period *t* is independent of the sequence of price changes during previous time periods. That is, knowledge of the sequence of price changes leading up to time period *t* is of no help in assessing the probability distribution for the price change during time period *t* "(Fama, 1965, 35). Nevertheless, perfectly independence can not be found in stock markets, thus to account for independence is just necessary that past history of the series would not allow prediction of the future in a way which makes expected profits greater than they would be under a buy-and-hold model. Fama in his discussion of independence concludes: "the stock market may conform to the independence assumption of the random walk model even though the process generating noise and new information are themselves dependent." (Fama, 1965, 39).

21 The normal distribution is also referred as Gaussian, because it was Karl F. Gauss (1977-1955) the one who introduced it when studying the motion of celestial bodies. (Jorion, 2007, 84)

22 "If price changes from transactions to transaction are independent identically distributed random variables with finite variance and if transactions are fairly uniformly spaced through time, the Central Limit theorem lead us to believe that price changes across differencing intervals such as a day week or month, will be normally distributed since they are the simple sum of the changes from transaction to transaction" (Fama, 1963, 297).

ties. First, the entire distribution can be characterized by its first two moments: the mean, which represents the location, and the variance, the dispersion. Second, "the sum of jointly normal random variables is itself normally distributed" (Jorion, 2007, 85).

Within this framework, a random walk is represented statistically by a *Brownian Motion*[23]. A Brownian movement can be defined as a stochastic process[24] that shares three basic properties: homogeneity in time, independence of increments and continuity of paths. Specifically, a Brownian motion is a process such that: 1) there is statistical stationarity, meaning that the process generating price changes stays the same over time[25]. Thus, if *Xt* denotes the process at $t>0$, the process $X_{t0} + t - X_{t0}$ has the same joint distributions functions for all $t>0$; 2) the increments of the process for distinct time intervals are mutually independent; and 3) the process declines in a continuous frequency, implying that most price changes are small and extremely few are large, and they change in a continuous manner. This excludes processes with sudden jumps, for instance. Therefore, this model assumes that small movements from $t_0$ to $t$ can be described as:

$$X_t - X_{t0} \approx e * |t - t_0|^H$$

where $e$ is a standard normal random variable and H = 0.5. In short, in a Brownian Motion, to be able to find $X_t$, a random number $e$ (chosen from a Gaussian distribution) is multiplied by the increment $|t - t_0|^H$, and the result is added to the given position $X_{t0}$.

## 2. Inconsistencies and Failures of Neoclassical Theory

While studying the performance of market prices, researchers have discovered certain market behaviors that contradict Neoclassical Theory. There is evidence for season anomalies[26],

23 The term "Brownian Motion" comes from the area of physics used to describe the irregular movement of pollen suspended in water, a phenomenon studied by the British physicist Robert Brown in 1928. Again, Bachelier was the first one to suggest that this process could also describe the price variations of financial series.

24 A stochastic process is determined by a deterministic component and a random variable, which can be only assessed by probability distributions.

25 For instance, tacking again the game of the coin-toss, statistical stationarity means that the coin itself does not change.

26 Researchers have found correlation of asset returns with market-book ratios, the firm size or even with the different seasons of the year. For more information refer to Fama (1965), Guimaraes, Kingsman and Taylor (1989), Lo (1997), and Lo and MacKinlay (1999).

for overreaction of investors, for an increase of correlation and excess volatility in certain periods of time, and for many other situations, which demonstrate that prices are not in their equilibrium values, investors do not act as rational individuals, and markets do not follow a random walk. This does not imply that there is something "abnormal" with the behavior of the financial market, but just that neoclassical postulates fail to describe the actual behavior of financial markets. As Lo (1997) explains: "What are we to make of these anomalies? Their persistence in the face of public scrutiny seems to be a clear violation of the EMH. After all, most of these anomalies can be exploited by relatively simple trading strategies." (Lo, 1997, xvi).

Several authors, such as Mirowski (1990, 2002, 2004), Hsieh and Ye (1998), Chorafas (1994), Peters (1994), Foster (2005), and Faggini and Lux (2009) among others, have highlighted that the discrepancy of economic and financial theory with reality rests in the fact that Neoclassical Theory resembles Newtonian physics. For instance, when Neoclassical Theory hypothesizes that the sum of rational individuals conduces the economy to an optimum equilibrium, it is possible to distinguish the Newtonian idea of modeling a linear system that exhibits a stable and well-defined "equilibrium".

Newtonian rigid determinism and timeless dynamics are also present in Neoclassical Theory (Mirowski 2004). In the Neoclassical realm, markets behave like a mechanical perfect roulette wheel, where no opportunities of arbitrage[27] are permitted. This mechanistic perspective allows the description of market' behavior with equations that can connect numerical measurements at a given time to their past and future values. Indeed, when statisticians hypothesized in the Random Walk Hypothesis that the course of a stock price follows a stochastic process such as a Brownian motion, they do not imply that prices cannot be forecasted[28]. In the contrary, "they merely imply that one cannot forecast the future based on past alone" (Cootner, 1964b, 80). For that reason,

27 Arbitrage refers to the possibility of profiting by exploiting price differences in the market without incurring in additional risk. In an efficient market, the competence between investors would not allow this possibility, therefore, it is said to be an arbitrage-free market.

28 "This is observed in Fama's work, who concluded in his 1965 research that: "it seems safe to say that this paper has presented strong and voluminous evidence in the random walk hypothesis", and then, introduced a paper with the sentence, 'there is much evidence that stock returns are predictable'" (in Jorion and Khoury, 1996, 326).

after the random variable, commonly referred as white noise, has been "separated out", deterministic equations can actually describe price changes. As Mirowski said: "These stochastic "shocks" had little or no theoretical justification, but themselves seemed only an excuse to maintain the pure deterministic ideal of explanation in the face of massive disconfirming evidence" (Mirowski, 2004, 231).

Besides Newton's concepts, classical mathematics, which involve linear systems with smooth and continuous changes, and symmetric distributions, are part of Neoclassical Theory[29]. In fact, the importance of this type of mathematical language is clearly reflected in the development of the theory. As it was mentioned before, the EMH was created to explain the random character of markets. In particular, it justified the use of statistical tools that required independence and Gaussian distributions. As Peters (1994) said: "the EMH, developed to make the mathematical environment easier, was truly a scientific case of putting the cart before the horse" (Peters, 1994, 41). Accordingly, the Newtonian mathematical idiom and reductionist methodology became essential for the Neoclassical approach to economics and finance[30].

Consequently, since the beginning, Neoclassical theorists refused to explain phenomena that contradicted their assumptions or could not fit in their mathematical equations (Hsieh and Ye, 1998). Therefore, large changes in prices or irrational behavior of individuals were just seen as been anomalous. Furthermore, it distanced them from the actual information revealed by real data (Mandelbrot, 2004). As a result, neoclassical doctrines are not grounded to empirical observation or even reasonable assumptions. As Mandelbrot (2004) observes: "the fact that mass psychology alone, might have been sufficient evidence to suggest there is something amiss with the standard financial models" (Mandelbrot, 2004, 170).

Overall, the failure of Neoclassical Theory is its chosen mode of

29 Just recall that a Brownian Motion describes the path of a stock in small independent changes that are distributed with the normal symmetric bell-shape curve.

30 This is argued by Chorafas (1994) and Foster (2005). Citing Foster (2005): "Why should eminently reasonable propositions concerning the existence of time irreversibility, structural change and true uncertainty in historical processes have been so unpalatable to the mainstream of the economics profession? Because of the chosen language of scientific discourse, namely mathematics. A scientific desire to use mathematics as formal medium for deduction. The problem does not lie in the chosen economics but, rather in its limited expression, in the chosen language of discourse". (Foster, 2005, 371)

discourse and set of tools. It took the ideas of the mid-19th century prior to the Second Law of thermodynamics, and remained mired in its original orientation even though the economic and financial world has changed enormously (Mirowski, 2002 and 2004). Therefore, to capture the complexity of the global economy and financial markets, it is necessary to renew our financial theories with the perspective of Chaos Theory and the Science of Fractals.

## B. Chaos Theory and the Science of Fractals in Finance

Both in physics and finance, the objective of Chaos Theory and the Science of Fractals is to study the aperiodic non-linear behaviour emerging from systems sensitive to the initial conditions that are part of a deterministic structure. Accordingly, the disordered behavior is a local property of the system, but there are in fact some distinguishable patterns of market behaviour. This is the main insight that this new paradigm gives to finance. It describes markets as having local randomness and global determinism, just as in fractal structures on nature.

A shift from an "efficient" market to a "fractal" market has certain implications. Financial systems that combine local randomness as well as global determinism cannot be explained by a random walk or normal distributions. Therefore in order to study these systems, it is necessary to find a new statistical description of capital markets. This will signify a change from Gaussian statistics to Fractal statistics.

### a) A Fractal Financial Market

Benoit Mandelbrot, father of fractal geometry, approached the market as a scientist, not a deductive mathematician, and in doing so, he was able to discover the self-similar property of financial markets. Indeed, he first discovered fractals in financial time series, when observing that the same kind of distributions appeared unchanged without characteristic scale. Weekly, monthly or yearly, it was possible to observe distributions with high peaks and "fat" tails that frequently followed a power of law[31]. For this reason, Mandelbrot concluded, "*the very heart of finance is a fractal*" (Mandelbrot, 2004, 165).

The importance of Mandelbrot's discovery is that it highlights that under the apparent disorder of capital markets, there are some "stylized facts" that can describe the behavior

31 This implies that graphs will not fall toward zero as sharply as a Gaussian curve.

of capital markets. For instance, in Mandelbrot's opinion, large and discontinuous price changes are far more common than what the Gaussian hypothesis predicts.

As observed in financial markets, transactions occur at different instants of time and are quoted in distinct units; hence, mathematically speaking a price series is never continuous. If prices move smoothly from one value to the other, it is possible to approximate the distribution to a normal. However, for Mandelbrot, prices are merely discontinuous as they go up or down very steeply, and even more, they tend to group. Thus, discontinuity is in fact a very common property of financial markets, and it is reflected in the "fat tails" of the distributions[32].

Dependence is also an important property for financial time series. In particular, long-term memory demonstrates that aleatory influences in the starting conditions play an important role in shaping the behavior of a dynamical system in the future. Therefore, contrary to the independence assumption of a random market, in a fractal market past events cannot be excluded.

### b) From a random walk to a multifractal process

Evidently, under these new assumptions of market behavior, it is not possible to represent variation of prices by the neoclassical random walk. Therefore, Mandelbrot (1997) proposed the *Fractional Brownian Motio*n (FBM), sometimes referred to as 1/f (fractional) noise. This stochastic process starts with the familiar Brownian motion: the distance traveled is proportional to the same power of the time elapsed[33]. However, in a fractional Brownian motion H[34] can range from zero to one allowing the process of price variations to describe the "wild" randomness of financial data. Because of the different fractional values that H can take, it is called a Fractional Brownian Motion. An $H = 0.5$ describes a Gaussian random process, an $H < 0.5$ means an anti-

32 In Mandelbrot's words: "Discontinuity far from being an anomalous best ignored, is an essential ingredient of markets that helps set finance apart from the natural science (…) [The only reason for assuming continuity] is that you can open the well-stocked mathematical toolkit of continuous functions and differential equations" (Mandelbrot, 2004, 86).

33 For a random fractal with a prescribed Hurst exponent, it is only necessary to set the initial scaling factor for the random offsets to
$\frac{1}{2} (|S + 1|^{2H} - 2|S|^{2H} - |1|^{2H})$ .

34 Here, H is referred to the Hurst Exponent.

persistent behavior[35], and an H > 0.5 is related to a persistent case[36].

The following "cartoons"[37] illustrate better the difference between a Brownian Motion and a Fractional Brownian Motion with different Hurts Exponents:

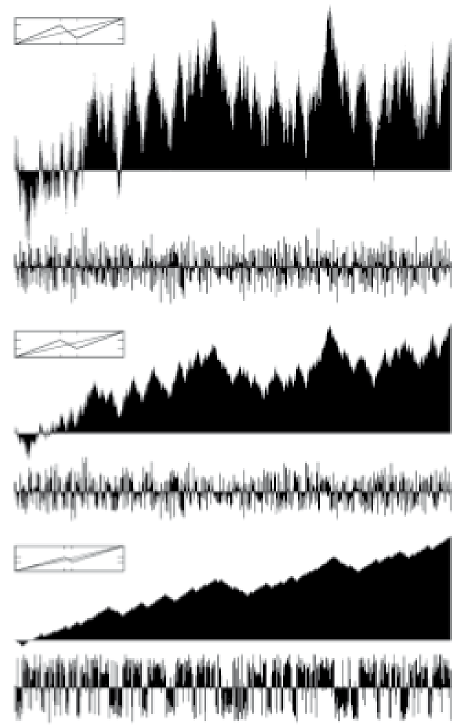

**Figure 2.1:** FBM with different Hurst exponents. The upper graph has an H=0.3, the middle graph has an H=0.5 and the bottom graph has an H=0.75 (Taken from Mandelbrot, 2005, 187).

Within this framework, two kinds of processes can be distinguished. A *uniscaling* or *unifractal* process, where its scaling behavior is determined from a unique constant *H*. This is indeed the case of a linear self-affine processes and, *H* is the self-affinity index or scaling exponent of the process. The other is a *multiscaling* process or *multifractal*, where different exponents characterize the scaling of different moments of the distribution. More precisely, it consists in letting the exponent *H* to depend on *t,* and to be chosen among infinity of possible different values. The key here is to introduce trading time[38], as price varia-

35 In this case, a positive trend is follow by a negative one, and a negative by a positive. For that reason, an antipersistent system covers less distance than a random process.

36 Long-term memory effects characterize a persistent behavior. In other words, what happens today influences subsequent changes or in chaotic terms, initial conditions affect future dynamics. This effect occurs regardless of time scale. For instance, daily returns are correlated with future daily price changes; as well as weekly changes are correlated with future weekly changes. For that reason, it is said that there is no characteristic time scale.

37 Mandelbrot named his graphs "cartoons" to avoid misunderstanding with what is known as models. He uses this term "in the sense of the Renaissance fresco painters and tapestry designer: a preliminary sketch in which the artist tries out a few ideas, and which if successful becomes a pattern for the full *oeuvre* to come" (Mandelbrot, 2004, 117).

38 Trading time is well defined in the stock exchange as the time that elapses during the open market hours. The introduction of this concept into financial models changes the Newtonian belief of absolute time to the relative concept of Einstein.

tions are best not followed in physical clock time but rather in trading time. "To implement this idea in a scaling world, one must identify price variations as a scaling function of trading time, and trading function as a scaling function of clock time"[39] (Mandelbrot, 1997, 55).

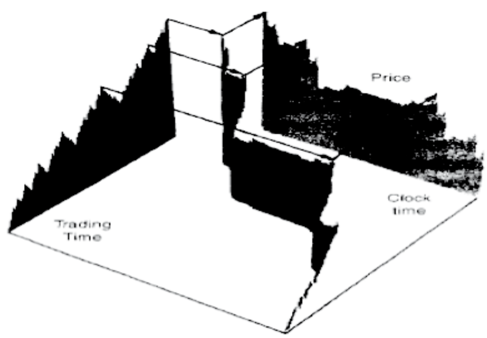


**Figure 2.2:** This 3D cube represents the "Baby Theorem". The right wall is the mother that provides a Brownian Motion in conventional clock time. The jagged line in the middle of the graph is the father, which deforms clock time into trading time. And the left wall is the multifractal baby measured in trading time. (Taken from Mandelbrot, 2004, 214)

Using Monte Carlo simulation[40], Mandelbrot was able to test the model in the computer. The result was a statistical similarity in the behavior of market prices. It was not completely identical, since the inputs were reduced to a smaller number of parameters, and thus, the outcome was undoubtedly affected. But as Mandelbrot explains:

> "In financial modeling all we need is a model "good enough" to make financial decisions. If you can distill the essence of GE's stock behavior over the past twenty years, then you can apply it to financial engineering. You can estimate the risk of holding the stock to buy your portfolio. You can calculate the proper value of options you want to trade on the stock. This is, of course, exactly the aim of all financial theory, conventional or not. The one difference: This time around, it would be nice to have an accurate model". (Mandelbrot, 2004, 221).

As it is observed, the importance of these models is that they take into account the "stylized facts" of financial markets, or in mathematical terms the "invariances", to statistically describe the real behavior market dynamics. Borland, Bouchard, Muzy and Zumbach (2005) characterized them as universal, in the sense that are common across different assets, markets and epochs. Similar to Mandelbrot insights, these authors found that em-

39 For a more detailed explanation refer to Mandelbrot (1997) and Calvet and Fisher (2002).
40 Broadly speaking, Monte Carlo simulations are numerical simulations of random variables made by advance computer techniques. They were first "developed as a technique of statistical sampling to find solutions to integration problems" (Jorion, 2007, 308). The importance of this method is that is an open-form technique, in the sense that it generates a whole distribution of possible outcomes, each of which allows the variables to migrate within predefined limits.

pirical data is characterized by certain qualitative properties: 1) the distribution of returns is in fact non-Gaussian, especially for short intervals of time that have a stronger kurtosis[41]; 2) volatility is intermittent and correlated what is known as volatility clustering; 3) Price changes scale anomalously with time ("multifractal scaling"). These are, indeed, not statistical irregularities, but the rules of market behavior.

Nevertheless, it is important to highlight that despite Mandelbrot's remarkable proposition, the search of a faithful financial model is not over yet. With Mandelbrot's investigations now it is possible to know that price changes behave very different from a random walk. But being such an undeveloped field, it is still subject to possible improvements[42].

## b) From Normal Distributions to Stable Paretian Distributions

For financial analysts, accepting Mandelbrot's ideas also means that the assumption of normal distribution is incorrect, as they do not account for discontinuity. For this reason, Mandelbrot proposed in his early work of 1960's to replace the Normal or Gaussian distribution assumption for the Stable Paretian Hypothesis. Mainly, stable Paretian distributions allow scaling properties or power law relationships.

Scaling distributions were first studied by Vilfredo Pareto, Italian economist, in his investigation of personal income in Italy. Pareto found that social classes were represented with a "social arrow" (not a pyramid), very fat at the bottom representing the poor mass of men, and very thin at the top describing the wealthy elite (Mandelbrot, 2004, 153-154). Pareto then modeled the wealth of individuals using the distribution $y = x^{-v}$, where $y$ is the number of people having income $x$ or greater than $x$, and $v$ is an exponent that Pareto estimated to be approximately 1.5. When calculating this relationship to other geographical areas, Pareto found that this result was also applicable to countries such as Ireland, Germany and even Peru.

Pareto's basic observation of a power of law was vey insightful. In

41 "*Kurtosis* describes the degree of flatness of a distribution" (Jorion, 2007, 87)

42 For instance, currently theorists are connecting multifractal Brownian motion with levy distributions to capture jumps, heavy tails, and skewness. This research was motivated since FBM are generated from Gaussian random variables, thus they have less power to capture heavy tails. They are also including GARCH processes and other techniques for improving forecast volatility. For more information of these developments please refer to Sun, Rachev and Fabozzi (2008).

his distribution of personal income Pareto involved tails that were heavy and followed a power-law distribution represented by $Pr(U>u) = u^{-\infty}$[43]. In this case, the probability of finding a value of *U* that exceeds *u* depends on $\alpha$. Such power laws are very common in physics and are a form of fractal scaling[44].

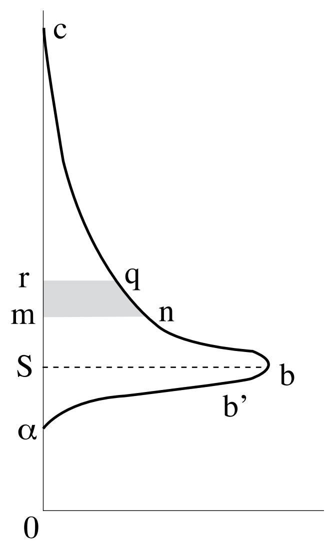


**Figure 2.3**: Pareto's 1909 diagram of the income curve (Taken from Mandelbrot, 2004, 154).

The long tailed distribution found in Pareto's work, led the French mathematician Levy, to formulate a generalized density function named Stable Paretian distribution, in which the normal and the Cauchy conform a special case[45].

These distributions can be described by four parameters: $\alpha$, $\beta$, $\delta$ and $\gamma$. The locational parameter is $\delta$, and if $\alpha$ is greater than one, $\delta$ will be equal to the expectation or mean of the distribution. The scale parameter is $\gamma$, and it can be compared to the measure of dispersion. Its value can be any positive number (Fama, 1963, 298 - 299). When $\gamma = 1$ and $\delta = 0$, the distribution is said to be in its reduced form.

Nevertheless, $\alpha$ and $\beta$ are the two parameters that determine the shape of the distribution. The parameter $\beta$ is an *index of skewness*[46] and must be in the interval between $-1 \leq \beta \leq 1$. When $\beta = 0$ the distribution is symmetric; if $\beta > 0$ the distribution is skewed right, and if $\beta < 0$ is skewed toward the left. On the other hand, $\alpha$ is the variable that describes the total probability

43 As applied to a positive random variable, the term scaling signifies scaling under conditioning. To condition a random variable U specified by the tail distribution $P(u) = Pr ( U > u)$. The power law is due to the characteristic exponent $\alpha$.

44 Fractals also scale also by a power law, more specifically, in fractals the range increases according to a power.

45 A Cauchy curve is a probability distribution with undefined mean, variance and other higher moments. Although it looks similar to a normal distribution, because of its symmetry and bell-shape curve, it has much heavier tails and higher peaks.

46 "*Skewness* describes departures of symmetry" (Jorion, 2007, 86).

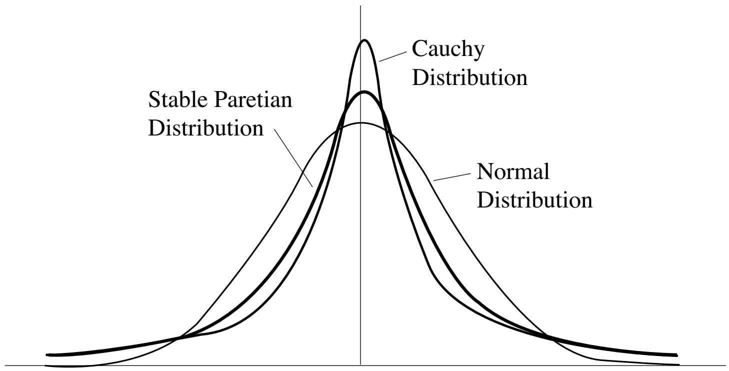


**Figure 2.4**: The three distributions: Normal, Cauchy and Stable Paretian. As it is observed, the former represents the intermediate curve between the Normal and the Cauchy Distribution (Taken from Mandelbrot, 2004, 40)

contained in the extreme tails. It is called the *index of stability* or *characteristic exponent*, and must be in the range from $0 < \alpha \leq 2$. When $\alpha = 2$, the distribution is normal and the variance exists. When $\alpha < 2$, there are more observations in the tails that under the normal distribution[47], and even more the variance becomes infinite or undefined[48].

Stable Paretian distributions have some desirable characteristics that allow them to describe the patterns of financial markets. First, they are invariant under addition meaning the sum of two Paretian distributions is itself stable Paretian[49]. This stability holds even when the values of the location and scale parameters, $\delta$ and $\gamma$, are not the same for each individual variable.

47 In other words, a small value of $\alpha$, implies thicker tails of the distribution.

48 Infinite variance implies that the variance of the time series changes over different samples and generally increases with the sample size. Therefore, it does not settle down to some constant value as it is assumed with normal distributions. This will entail that the sample variance is not statistically significant. If $\alpha \leq 1$, the same will happen for the mean as it would not exist in the limit. Nevertheless, it is important to point out that as in all fractal structures, there is eventually a time frame where fractal scaling ceases to apply, and thus there could be a sample size where variance does indeed become finite. But within our life time (at least 100 years), stable distributions will behave as if they have infinite variance.

49 Indeed, this is the reason why they received the name stable. Stable means that the basic properties of an object remain unaltered even though it is rotated, shrunk, or even add it to something else.

Second, they allow an asymmetric representation of the distribution of returns with high peaks and fat tails. As a result, with these distributions it is possible to model abrupt and discontinuous changes. Examples of this behaviour are found in market critical dynamics that amplify the bullish or bearish sentiment.

These distinctive characteristics led Mandelbrot (1963a) to propose the *Stable Paretian Hypothesis* arguing that "1) the variance of the empirical distribution behave as if they were infinite; 2) the empirical distributions conform best to the non-Gaussian member of a family of limiting distributions called stable Paretian". (Fama, 1963, 298). His basic idea is to model the percentage changes in a price as random variables with mean zero, but with an infinite standard deviation. In other words, the distribution of speculative prices is defined by the interval $1 < \alpha < 2$, contrary to the Gaussian hypothesis that states that $\alpha = 2$.

For the moment, Mandelbrot's hypothesis cannot be taken as definite. As it is observed, this statement was proposed in the early 1960's, and at that time, it received support for a small group of economists, including Eugene Fama. However, with the release of the famous survey of Fama in 1970 about *efficient markets*, the academy disregarded Mandelbrot's idea in favor of the Gaussian assumption. Therefore, empirical evidence is not so extensive in this topic, and it is still necessary to prove if indeed $\alpha$ ranges in the interval $1 < \alpha < 2$. It could take lower or higher values, for instance[50].

However, the idea that distributions of returns are in fact non-Gaussian deserves more attention. This proposition is important because it highlights that there are more large abrupt changes, and hence, markets are inherently more risky than those described by a Gaussian market. For this reason, some financial researchers have started to investigate the validity of this assumption in risk models. Rachev, Schwartz and Khindanva (2003) present one of the most complete investigations on the measurement

50 The empirical analysis of testing the Stable Paretian Hypothesis will not be included in this article. Even so, it is important to highlight that some researchers have found that indeed $\alpha$ deviates from the Gaussian case of 2. One of the most recent studies is Johnson, Jefferies and Ming Hui (2003), which investigated the composite index recorded in a daily basis between 1966 and 2000, and the Shangai stock exchange index recorded at 10-s intervals during the period of eight months in 2002. In this study, they found that alpha equals to 1.44 for both markets. Although this value can change from market to market, it does prove that markets have scaling properties.

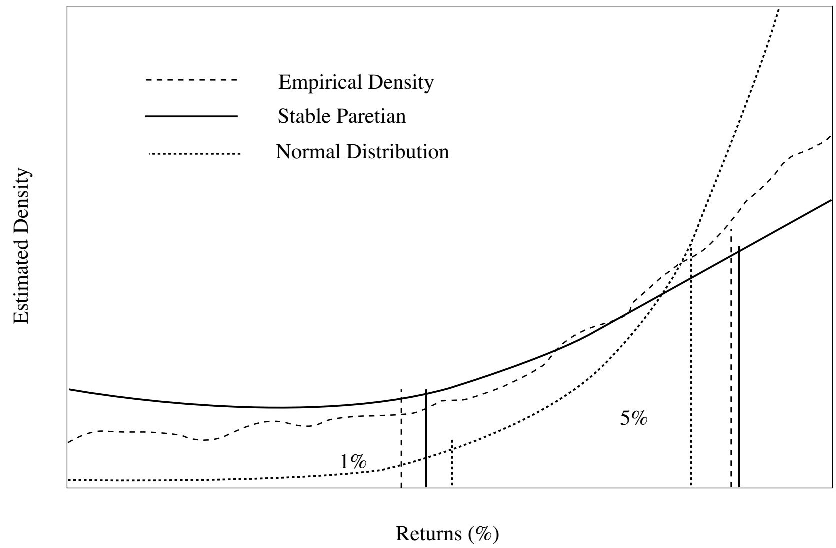


**Figure 2.5:** Representation of Rachev, Schwartz and Khindanva (2003) studies about normal-VaR with stable Paretian-VaR.

of Value at Risk (VaR)[51] with stable paretian distributions. These authors compared the results of normal-VaR with stable Paretian-VaR, concluding the following:

1) The stable modeling generally results in a more conservative and accurate 99% VaR estimate than the one made with the normal distribution assumption. In fact, the normal distribution leads to overly optimistic forecasts of losses in the 99% quantile.

2) With respect to the 95% VaR estimation, the normal modeling is acceptable from a conservative point of view. The stable model underestimated the 95% VaR, but the estimate was actually closer to the true VaR than the normal estimate.

Harmantzis, Miao and Chien (2005) found similar results[52]. For VaR estimation at a confidence level of 99% heavy tails models, such as the stable Paretian, produced "more ac-

51 VaR is a risk measure defined as the worst loss over a target horizon that will not be exceeded with a given level of confidence. If *c* is the confidence level, then VaR corresponds to the *1- c* lower tail level of the distribution of returns. Formally, it can be defined as follows: P ( Loss > VaR) = 1 - c. The choice of confidence level and time horizon depends on the purpose of VaR. However, the confidence level is typically between 95% and 1%. Regulation (Basel I Accord) recommends a 99% confidence level in a 10 days horizon.

52 The data series included four currency exchange rates: USD/Yen, Pound/USD, USD/Canadian, USD/Euro; and six stock market indices: S&P500 (US), FTSE100 (UK), Nikkei225

curate VaR estimates than non-heavy tailed models, especially for data that exhibit heavy tails" (Harmantzis, Miao and Chien, 2005, 9). However, in the 95% confidence level, Gaussian models resulted in more accurate VaR results. Again, Ortobelli, Rachev and Fabozzi (2009) concluded the following[53]: "The empirical evidence confirms that when the percentiles are below 5%, the stable Paretian model provides a greater ability to predict future losses than models with thinner tails" (Ortobelli, Rachev and Fabozzi, 2009, 16).

As it is observed, empirical evidence demonstrates that in the 99% quantile VaR estimates with stable Paretian distributions are more accurate than assuming normal distributions. This implies a significant improvement in risk management' models, as the risk in the extremes of the distributions can be measure more adequately.

Conclusively, accepting Mandelbrot's hypothesis implies that current models based on the normal distribution are misleading, as they do not account for the real risk in financial markets. In fact, this will mean a change in the assumptions behind the Black and Sholes models for pricing options, the Merton model for pricing credit risk, and, as it was demonstrated, Value-at-Risk. Almost all of the models that Wall Street uses to make financial decisions nowadays would need to be seriously revaluated. As Cootner, MIT economist said in his review of the Mandelbrot (1963b): "Mandelbrot, like Prime Minister Churchill before him, promises us not utopia but blood, sweat, toil and tears" (Cootner, 1964d, 337). Stable Paretian distributions, however, seem to be a robust assumption as they account for asymmetry, and most important, for the "inconvenient" outliers.

## Conclusion

Chaos Theory and the Science of fractals have already demonstrated the great progress it has brought to science. In physics, when scientists left the Newtonian vision of the world to observe its real complexity and roughness, they were able to have a better comprehension of natural systems. For this reason, the advances that were done in this field motivated other disciplines to take the same step forward.

(Japan), DAX (Germany), CAC40 (France), TSE300 (Canada). The data covered a period from January 1990 to December 2003; DAX started in January 1991; CAC40 in March 1991, and Euro/USD began in January 1999.

53 The study of these authors was made in the MSCI WorldIndex, a stock market index of 1500 "world" stocks. The index includes stock from emerging markets.

For instance, the inclusion of Chaos Theory in economics (*complex economics*) has allowed the exploration of economic phenomena with a more proper synthesis. Now, economists do not have to justify equilibrium, rationality and linearity; but instead they can address the intricate behavior of economic reality.

Nevertheless, in both fields, it has been the effort of dissatisfied academics that have triggered a change in mainstream theories. In finance, this audacious work has been done for just a few, who have been ignored and have not received the proper attention. It seems like financial analysts refuse to leave their old scientific methods of inquiry to embrace a new paradigm more in accordance with the new science. As a result, the ideas and method of Newton's time, reflected in Neoclassical theory, continue to be deeply rooted in the financial industry, even so the world has changed enormously.

This slow advance in financial theory has come with a very high price. Just in the last 20 years, it is possible to observe how financial crisis have augmented in number, size and value. Each one has struck the financial sector harder and in a more global scale. But in our current financial models, these events should have never happened. They were so improbable that they were just considered far far far outliers. Classical models simply fail to recognize the increasing complexity of financial markets, and consequently, they have led finance analysts to serious estimation errors.

For this reason, to be able to cope with the challenges of this new era, it is necessary to move away from the neoclassical approach to finance. This thesis proposes a fractal view of the market, as until now, it provides a more adequate perspective to understand financial behavior. It recognizes its inefficiency and irrationality, and most important, it emphasizes it roughness. Consequently, this new paradigm would allow professionals in this area to work with the adequate vocabulary and method to address today's capital markets. For risk managers, in particular, it would imply better models that can increase their awareness of the risky nature of markets. Thus, they would finally be able to receive clear warnings when trouble is ahead, and allow them to be better prepared. Perhaps, by adopting Chaos Theory and the Science of Fractals in finance, the next crises can truly come with a "discounted" value for financial institutions.

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