The origin of present value as a capital budgeting decision criteria: A journey to Pisa in the middle ages

El origen del valor presente como un criterio para la decisión de presupuesto de capital: un viaje a Pisa en la Edad Media

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Abstract

The emergence of new techniques for capital budgeting decision criteria, such as real options, have not managed to displace net present value as one of the most used methodologies to evaluate the viability of an investment. This article seeks to explore the origins of present value as a technique for evaluating investment alternatives. William Goetzmann places the origin of this methodology in the Middle Ages, specifically in the first decades of the thirteenth century, in *Liber abbaci*, the masterpiece of Leonardo Pisano, better known as Fibonacci. The previous reassesses the belief that places this concept in Irving Fisher's work of 1930. As to the origin of the nickname of Fibonacci, this investigation found that Pietro Cossali used and explained its meaning in his writing of 1799. This therefore, allows for reevaluating the belief that the nickname of Fibonacci was given by Guillaume Libri in 1838. Finally, this document offers a tribute to all those researchers, in this case of the history of mathematics, who because of their archaeological work rediscover advances and developments that otherwise would be forgotten.

Key words: History of economic thought; Leonardo Pisano (Fibonacci); capital budgeting criteria (net present value); Pisa.

JEL classification: B10, B31, G11, N83

Resumen

El surgimiento de nuevas técnicas para la evaluación de inversiones, como las opciones reales, no ha logrado desplazar al valor presente neto como una de las metodologías más utilizadas para evaluar la viabilidad de una inversión. Este artículo busca explorar los orígenes del valor presente neto como una técnica para la evaluación entre alternativas de inversión. William Goetzmann sitúa el origen de esta metodología en la Edad Media, específicamente en las primeras décadas del siglo XIII, en *Liber abbaci*, la obra maestra de Leonardo de Pisano, más conocido como Fibonacci. Lo anterior reevalúa la creencia que sitúa este concepto en los trabajos de Irving Fisher de 1930. En cuanto al origen del sobrenombre de Fibonacci, esta investigación encontró que Pietro Cossali utilizó y explicó su significado en un escrito de 1799. Lo anterior permite reevaluar la creencia de que el apodo de Fibonacci fue dado por Guillaume Libri en 1838. Por último, este documento ofrece un tributo a todos aquellos investigadores, en este caso de la historia de las matemáticas, que como consecuencia de su trabajo arqueológico redescubren avances y desarrollos que de otra forma serían olvidados.

Palabras clave: historia del pensamiento económico; Leonardo de Pisa (Fibonacci); criterios de decisión de inversión (valor presente neto); Pisa.
Clasificación JEL: B10, B31, G11, N83

Introduction

According to Barone (2008), who cites Goetzmann (2004), Leonardo Pisano (ca. 1170-1250), Italian mathematician, better known as Fibonacci, is at the base of the tree of financial economics, not for the diffusion of Arabic numbers in the West, but rather because he was "the first to develop present value analysis for comparing the economic value of alternative contractual cash flows" (Barone, 2008, p. 4).

As Rubinstein (2006) notes, the academic community placed the origin of the present value concepts with Fisher (1930), an author who collected previous developments in Fisher (1896), Fisher (1906) and Fisher (1907). In the article of 1896, Fisher expressed that "the literal meaning of 'present value' implies that it is the actual market price today of a future sum due" (Fisher, 1896, p. 361). Subsequently, in 1907, Fisher indicated that "for, of various optional employments of his capital, the investors select the one which offers the maximum present value" (Fisher, 1907, pp. 24-25). In addition, Rubinstein (2003) highlighted the concept of *present value* as one of the *Great Moments in Financial Economics*. In that publication, Rubinstein identified Johan de Witt's work published in 1671, under the title *Value of Life Annuities in Proportion to Redeemable Annuities*, as one of the precursors to the definition of the present value concept.

However, because of Goetzmann's (2004) work, the roots of the present value concept go back to the beginning of the thirteenth century, specifically to Pisano, in his work *Liber abbaci*, the first edition of which dates from 1202. Due to the foregoing, it is concluded that this is a new situation of Stephen Stigler's law of eponymy (Rubinstein, 2006). As Barone (2008) states, Rubinstein (2006) relates at least nineteen cases of Stigler's Law of eponymy in finance, one of them being the present value concept.

In addition, despite the emergence of new methodologies for capital budgeting decisions, such as real options, several research papers—Bennouna, Meredith & Marchant (2010); Ross, Westerfield & Jaffe (2013); Brunzell, Liljeblom & Vaihekoski (2013); Wnuk-Pel (2014); Andor, Mohanty & Toth (2015); Souza & Lunkes (2016) and Afeera Mubashar (2019)—show that net present value technique remains one of the most popular methodologies for investment decision making.

This article aims to review in *Liber abbaci*, the way that Pisano on the one hand, developed what today in financial mathematics is identified as *equivalence relations*, and on the other hand, the way he used this to introduce *present value* concept, as a capital budgeting technique.

To carry out the proposed objective, this article is divided into five parts, this being the first. In the second section there is a brief review of the work *Liber abbaci*, a book in which Pisano introduced among others not only the *present value* concept but also the series known as *Fibonacci numbers* or *Fibonacci sequence*. Next, the third part details the way Pisano presented and developed both concepts of equivalence and present value, the latter as a tool to evaluate different investment alternatives. The fourth section is related to the rediscovery of Pisano's work. It was necessary to wait eight centuries so that firstly, there was a translation of *Liber abbaci* from Latin to a modern language, like English (Sigler, 2003), and secondly, recognize the degree of development of financial mathematics of the thirteenth century, to the point that Goetzmann (2004), in the article in which he presented his financial findings on *Liber abbaci*, gives it the title of *Fibonacci and the Financial Revolution*. The fifth and last part, shall present some conclusions.

1. *Liber Abbaci*

Due to the relevance of *Liber abbaci*, the beginning of this work is presented. In it, Pisano showed not only the Indian numbers to which they add the zero and explained that with them it is possible to express any number, but he also presented the principles of numerical position:

> Chapter 1. Here Begins the First Chapter
> The nine Indian figures are:
> 9 8 7 6 5 4 3 2 1
> With these nine figures, and with the sign 0 which the Arabs called zephir any number whatsoever is written, as is demonstrated below. A number is a sum of units, or a collection of units, and through the addition of them the numbers increase by steps without end. First, one composes from units those numbers which are from one to ten. Second, from the tens are made those numbers which are from ten up to one hundred. Third, from the hundreds are made those numbers which are from one hundred up to one thousand. Fourth, from the thousands are made those numbers from one thousand up to ten thousand, and thus by an unending sequence of steps, any number whatsoever is constructed by the joining of the preceding numbers. The first place in the writing of the numbers begins at the right. The second truly follows the first to the

> left. The third follows the second. The fourth, the third, and the fifth, the fourth, and thus ever to the left, place follows place. And therefore the figure that is found in the first place represents itself; that is, if in the first place will be the figure of the unit, it represents one; if the figure two, it represents two; if the figure three, three, and thus In order those that follow up to the figure nine; and indeed the nine figures that will be in the second place will represent as many tens as in the first place units; that is, if the unit figure occupies the second place, it denotes ten; if the figure two, twenty; if the figure three, thirty; if the figure nine, ninety.
>
> And the figure that is in the third place denotes the numbers of hundreds, as that in the second place tens, or in the first units; and if the figure is one, one hundred; if the figure two, two hundred; if the figure three, three hundred, and if the figure nine, nine hundred, Therefore the figure which is in the fourth place donates as many thousand as in the third, hundreds, and as in the second, tens, or in the first, units; and thus ever changing place, the numbers increases by joining. (Sigler, 2003, pp. 17-18)

According to Devlin (2017), Pisano published two editions of *Liber abbaci*. The first one in 1202, and the second one in 1228. Of the 1202 edition, there is no existing copy. There are fourteen copies of the 1228 edition. Of these copies, three are complete or almost complete and all of them are in Italy. The first one, is in *Biblioteca Apostolica Vaticana* (Vatican Apostolic Library), better known as the Vatican Library in Rome, albeit an incomplete copy because chapter 10 is missing. The second copy is in the *Biblioteca Nazionale Centrale di Firenze* -BNCF- (Florence National Central Library), a copy complete, and the third one, is in *Biblioteca Communale di Siena* (Siena Public Library), an incomplete copy because it is missing much of chapter 15. The copy that is in BNCF was used by Baldassarre Boncompagni (1921-1894), an Italian mathematics historian, who published the first reedition of Pisano's *Liber abbaci* in the mid-nineteenth century, exactly in 1857 (Boncompagni, 1857). It is important to mention that this copy was made in the same language in which *Liber abbaci* was originally written, Latin. Boncompagni's edition was the basis for the translation of *Liber abbaci* to English (Sigler, 2003), eight centuries after the first edition of *Liber abbaci* was published by Pisano in 1202. About the remaining eleven copies, eight are in Italy and three in France. The eight existing in Italy are distributed as follows: four in BNCF, and the remaining four, in *Biblioteca Laurenziana* (Laurentian Library) in Florence, *Biblioteca Riccardiana* (Ricardian Library) in Florence, *Biblioteca Ambrosiana* (Ambrosian Library) in Milan and *Biblioteca Nazionale Centrale* (National Central Library) of Naples. The three copies that are in France, specifically in Paris, one is in *Bibliothèque Mazarine* (Mazarin Library) and two are in *Bibliothequè Nationale de France* (National Library of France) (Devlin, 2017).

At this point, a small reflection about the title of *Liber abbaci*. Its correct translation is *Book of calculation* and not *Book of the abacus*. Pisano's work showed that arithmetic could be done without using the abacus. Devlin (2011, pp. 11-12) showed the origin of this confusion:

> The distinction is reflected in Leonardo's spelling. The Latin and Italian word *abbacus* was used in medieval Italy from the thirteenth century onward to refer to the method of calculating with the Hindu-Arabic number system. The first known written use of the word *abbacus* with the spelling and meaning was in fact in the prologue of Leonardo's book. Thereafter, the word *abbaco* was widely used to describe the practice of calculation. A *maestro d'abbaco* was a person who was proficient in arithmetic. In fact, *abbaco* still has that as its primary (preferred) meaning in contemporary Italian.
>
> Medieval authors did not usually give their works titles. The name we used today for Leonardo's book comes from his opening statement:
>
> *Here begins the Book of Calculation*
> *Composed by Leonardo Pisano, Family Bonacci,*
> *In the Year 1202*.

Liber abbaci is divided into 16 sections: Dedication and Prologue and fifteen chapters[1]. Table 1 presents the percentage that each section has in the English version of *Liber abbaci* (Sigler, 2003). As you can see, the five extended sections represent 68.4 percent of the book, and the longest section corresponds to Chapter 12, which is titled *Here Begins Chapter 12*.

Table 1: *Liber abbaci's* chapter detail

Chapter No.	Detail	Pages	No. of Pages	%
	Dedication and Prologue	15-16	2	0.3
1	Here Begins the First Chapter	17-22	6	1.0
2	On the Multiplication of Whole Numbers	23-38	16	2.7
3	On the Addition of Whole Numbers	39-43	5	0.8
4	On the Subtraction of Lesser Numbers from Greater Numbers	45-47	3	0.5
5	On the Divisions of Integral Numbers	49-76	28	4.7
6	On the Multiplication of Integral Numbers with Fractions	77-98	22	3.7

1 For a detailed description of the contents of each chapter of *Liber abbaci*, the interested reader can review Sigler (2003), Devlin (2011) and Devlin (2017).

Chapter No.	Detail	Pages	No. of Pages	%
7	On the Addition and Subtraction and Division of Numbers with Fractions and the Reduction of Several Parts to a Single Part	99-126	28	4.7
8	On Finding the Value of Merchandise by the Principal Method	127-177	51	8.6
9	On the Barter of Merchandise and Similar Things	179-211	33	5.6
10	On Companies and Their Members	213-226	14	2.4
11	On the Allowing of Monies	227-257	31	5.2
12	Here Begins Chapter Twelve	259-445	187	31.5
13	On the Method Elchataym and How with It Nearly All Problems of Mathematics Are Solved	447-487	41	6.9
14	On Finding Square and Cubic Roots, and on the Multiplication, Division, and Subtraction of Them, and On the Treatment of Binomials and Apotomes and their Roots	489-530	42	7.1
15	On Pertinent Geometric Rules And on Problems of Algebra and Almuchabala	531-615	85	14,3
Total				**100.0**

Source: Prepared by the author from Liber abbaci (Sigler, 2003).

2. Present value: A middle age concept

Goetzmann (2004) classified the financial problems presented in *Liber abbaci* in the following four types: a) division of profits, b) traveling merchant profits, c) interest rate and banking problems, and d) present value analysis. While the first is in Chapter 8, the remaining three are in Chapter 12. Taking into consideration that the last three problems involve the concept of value of money over time, a detailed analysis of each of them is performed.

According to Table 1, Chapter 12 titled simply *Here Begins Chapter Twelve* is the *Liber abbaci's* longest chapter, with an extension of 31.5 per cent. As can be seen in Table 2, Chapter 12 had nine parts. The concepts of equivalence and present value concept, are developed in part 6, titled *On Problems of Travellers and also Similar Problems*. By extension, this part occupies 23 pages of *Liber abbaci* (Table 2) and represents 12.3 per cent of Chapter 12.

Table 2: *Liber abbaci's*, Chapter 12. *Here Begins Chapter Twelve detail*

Part No.	Detail	Pages	No. of Pages	%
1	On Summing Series of Numbers	259-263	4	2.1
2	On Proportions of Numbers	263-268	5	2.7
3	On Problems of Trees and Other Similar Problems, for Which Solution Are Found	268-317	49	26.2
4	On the Finding of a Purse	317-337	20	10.7
5	On the Purchase of Horses among Partners According to Some Give Proportion	337-372	35	18.7
6	On Problems of Travellers and also Similar Problems	372-395	23	12.3
7	On the Method of False Position for Two Man Who Ship Wool for a Fee	395-427	32	17.1
8	On Certain Divinations	427-435	8	4.3
9	On a Series of Powers of Twos on Chessboard Squares and Some Other Methods	435-445	11	5.9
Total			**187**	**100.0**

Source: Prepared by the author from Liber abbaci (Sigler, 2003).

2.1. Traveling merchant profits

The first exercise that Pisano proposed in this part seeks to establish, in contemporary terms, the present value of a constant annuity. The annuity is represented by the expenses that a traveler has in the different cities he visited. In Pisano's example, this amounts to 12 denari. With respect to the term of the operation, Pisano assimilated the trip between cities with the passage of time. In this first example, the traveler went from Pisa to Lucca, then from Lucca to Florence, and finally, from Florence to Lucca, in total he made three trips equal to three years. Regarding the cost of money, Pisano pointed out that the traveler in each city doubles his money. This leads to a one hundred percent return on investment. In Pisano's words the problem:

A certain man proceeding to Lucca on business to make a profit doubled his money, and he spent there 12 denari. He then left and went through Florence; he there doubled his money, and he spent 12 denari. Then he returned to Pisa, doubled his money, and spent 12 denari, and it is proposed that he had nothing left. It is sought how much he had at the beginning. (Sigler, 2003, p. 372)

When Pisano presented the solution to this first traveler's problem, he began to establish the discount factor. Since the exercise indicated that in each trip (one year) the traveler duplicates the money, it is relevant to point out that one denari in the present time is equivalent to two in the future, so the discount factor is equal to $\frac{1}{2}$. Considering the above, Pisano began to discount the different cash flows, which in this case are equal to 12 denari, starting with the last one. As indicated, the third flow is in year 3, therefore, it is necessary to discount it three times, equal to $\frac{1}{8}\left(\frac{1}{2}*\frac{1}{2}*\frac{1}{2}\right)$. The second cash flow that is in year 2, must be discounted twice. In this way, a discount rate of $\frac{1}{4}\left(\frac{1}{2}*\frac{1}{2}\right)$. The first cash flow, which is in year 1 is discounted once, equal to $\frac{1}{2}$. Since the value of the annuity is the same for the three cash flows, Pisano proceeded then to add the three discount factors, previously found, $\frac{1}{8}+\frac{1}{4}+\frac{1}{2}$, and obtained a discount factor for all the annuities, equal to $\frac{7}{8}$. At this point, to determine the present value of the annuity, Pisano multiplied the value of the annuity (12 denari), by the numerator of the discount factor (7), and as a result, the present value equal to $\frac{84}{8}$, equivalent to $10\,\frac{1}{2}$ denari. Pisano called this methodology of problem solving, as *travellers*. Next Pisano's solution:

Because it is proposed that he always doubled his money, it is clear that 2 will be made from one. Whence it is seen what fraction 1 is of 2, namely $\frac{1}{2}$, which thus is written three times because of the three trips that he made: $\frac{1}{2}\,\frac{1}{2}$, and the 2 is multiplied by the 2 and the other twos that are under the fraction; there will be 8 of which you take $\frac{1}{2}$, namely 4, of which you take $\frac{1}{2}$, namely 2, and of the 2 you take $\frac{1}{2}$, namely 1. After this you add the 4 to the 2 and the 1; there will be 7 that you multiply by the 12 denari which he spent; there will be 84 that you divide by the 8; the quotient will be $\frac{1}{2}$ 10 denari and the man had this many. For example, he doubled the $\frac{1}{2}$ 10 denari making 21, of which he spent 12 leaving 9; this he doubled making 18 of which he spent 12 leaving 6; it again he doubled making 12, from which subtracting the expense, namely the 12, nothing remains, as was proposed, and thus you will be able to operate with IIII or more trips. (Sigler, 2003, pp. 372-373)

After presenting the solution of this problem, Pisano made a slight modification to it and proposed to establish the present value, under the assumption that at the end of the last trip (end of term), there will be a surplus of 9 denari.

From the previous result it is known that the present value of an annuity composed of three equal cash flows, of 12 denari each, is equal to $\frac{84}{8}$, so it is necessary to add the present value of a cash flow that is in year 3, and whose value is equal to 9 denari. As was seen in previous form, its discount factor is equal to $\frac{1}{8}$, so the present value of this is equal to $\frac{9}{8}$. In this way, to find the present value of the annuity that has an extraordinary flow at the end of the term, it is enough to add the present value of each flow, that is $\frac{84}{8}+\frac{9}{8}$, that gives a result of $\frac{93}{8}$, equal to 11 $\frac{5}{8}$ denari. Next, Pisano's solution:

However if it is proposed that in the last of the aforewritten trips some denari remain after the expenditures, we say 9, then the 9 is added to the 84 found above; there will be 93 denari which is divided by the 8, as we said before; the quotient will be $\frac{5}{8}$ 11, and he had this many (Sigler, 2003, p. 373).

Then Leonardo presented a series of problems in which, for example, the objective is to find the annuity given both a present and future value, and an interest rate.

2.2. Interest rate and banking problems

In the following example, which is identified as *On a Man Who Invests for Interest without Notice*, Pisano presents the case of an investor who deposits an unknown amount of money at interest (20 per cent per annum) for five years and seventy days. Every year and at the expiration of the term, he withdraws some resources, which he identifies as a dividend (30 denari annually). Unlike other examples, where the term is expressed in a whole number of years, in this situation, the investment term amounts to five years and seventy days, that is, a total of 1870 days. For its solution, Pisano presented two methods. While in the first, he applied the methodology that he called *travellers*, in the second, he introduced the method that he called the *reverse method*. In this last technique, Pisano began by discounting the last cash flow, that as noted was placed in year 5 plus 70 days to year 5, and then that value was added to the cash flow (dividend) of year 5 and discounted to year 4, and so on until the value in present time is reached, and this value corresponds to the value of the initial investment. Next, the text of *On a Man Who Invests for Interest without Notice* and the solution offered by Pisano, using *travelers'* method:

> Also a certain man invests denari, I know not how many, at the same interest, and how many must he give for a dividend of 30 pounds per year in the same house. He indeed holds in the house the denari for 5 years and 70 days. The amount of denari is sought. Beginning first with 70 days, namely in order that you see how many denari he must hold 70 days in the house. And it will be seen so: because the interest of the first year is $\frac{1}{5}$ of the total capital one must multiply the days of the year by 5; there will be 1800 to which you add the 70 aforewritten days; there will be 1870; therefore in the 70 days 1870 is made from 1800, that is 187 is made from 180; therefore you put the 180 over the 187 thus, $\frac{180}{187}$; next you see how much is the dividend of the 70 days thus; you multiply the 30 by the 70, and you divide by 360; there result $\frac{5}{8}$ 5 pounds for the dividend of the 70 days; you multiply this by the 180, and divide by the 187; the quotient will be $\frac{115}{187}$ 5 pounds; all of this is explained thus; you will be able to include this under the trip method, namely for the 5 years you say five trips. In each of the 6 is made from 5, and 30 pounds are spent in each trip, namely the dividend; and at the end of the 5 trips, that are 5 years, there remain $\frac{115}{187}$ 5 pounds which he holds in the house 70 days; therefore as we taught above, the $\frac{5}{6}$ is written down five times in order thus: $\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6}$; next you will multiply the 6 by the 6, and the 6, and the 6, and the 6; namely by all of the numbers which are under the fractions; there will be 7776 of which you take $\frac{5}{6}$ that is 6480; of it you take $\frac{5}{6}$ that is 5400; of this you take $\frac{5}{6}$ that is 4500; of this you take $\frac{5}{6}$ that is 3750; of this you take $\frac{5}{6}$ that is 3125; next you add the 6480, 5400, 4500, 3750, and 3125; there will be 23255 that you multiply by the 30 pounds dividend; there will be 697650. Also you multiply the 3125 by the $\frac{115}{187}$ 5, and the product that results you add to the 697650, and you divide the sum with the rule for the 7776 that is $\frac{1\ 0\ 0\ 0\ 0}{6\ 6\ 6\ 6\ 6}$; the quotient will be $\frac{5\ 1\ 6\ 1\ 6}{6\ 6\ 11\ 17}$ 91 pounds, and this many are the amount of the denari that he invested. (Sigler, 2003, p. 388)

The first calculation that Pisano did was to find the value of the last payment (dividend) that is placed on day 1870. For the examples before the case *On a Man Who Invests for Interest without Notice*, he knew that the annual interest rate was 20 per cent, or in his terms, he obtained six denari from five, or five denari give one. According to this, an annual dividend of 30 denari, is equivalent to a dividend of $5\,\frac{5}{6}$ for a period of 70 days $\left(30 * \frac{70}{360}\right)$. Then Pisano determined the appropriate rate of discount for 70 days. The interest for 70 days equals to $\frac{7}{180}$ (the result of multiplying the annual interest rate of 20 per cent or $\frac{1}{5}$ by 70 days and the result, divided by 360 days). In this way, the discount rate is equal to $\frac{180}{187}\left(\text{inverse of } 1+\frac{7}{180}\right)$. When you applied this factor of discount $\frac{180}{187}$ to the cash flow on day 1870, that is $5\,\frac{5}{6}$, it results in a present value at fifth year of $5\frac{115}{187}\left(5\frac{5}{6} * \frac{180}{187}\right)$. At this point, the problem is reduced to discounting to present time (year 0) the following six cash flows: from year 1 to year 5 an annual and equal cash flow of 30 denari, and in year 5 a cash flow of $5\,\frac{115}{187}$ pounds. To determine the discount rate to be applied to each one of the annual cash flows, it is enough to remember that an investment of 5 denari in the present time, becomes 6 denari, in a term of one year, from which, the annual discount rate is equal to $\frac{5}{6}$. At this point, to find the present value of the cash flows, Pisano divided the problem into two stages: in the first, the five equal annual cash flows each one of 30 denari discounted to present value, identified as P_1, to which he then added the present value of the additional cash flow of year 5, this is $5\,\frac{115}{187}$ denari equal to $\frac{1.050}{187}$, identified as P_2.

Then, the present value of the five equal annual cash flows is equal to:

$$P_1 = 30 * \frac{5}{6} + 30 * \frac{5}{6} * \frac{5}{6} + 30 * \frac{5}{6} * \frac{5}{6} * \frac{5}{6} + 30 * \frac{5}{6} * \frac{5}{6} * \frac{5}{6} * \frac{5}{6} + 30 * \frac{5}{6} * \frac{5}{6} * \frac{5}{6} * \frac{5}{6} * \frac{5}{6}$$

$$P_1 = 30 * \left[\frac{5}{6} + \frac{25}{36} + \frac{125}{216} + \frac{625}{1.296} + \frac{3.125}{7.776}\right] = 30 * \left[\frac{23.255}{7.776}\right] = \frac{697.650}{7.776}$$

Then, the present value of the last cash flow that is in year 5, equal to $\frac{1.050}{187}$:

At this point, to find the present value sought, it is enough to add to the present value of the annual series (P_1) the present value of the extra cash flow (P_2).

$$P = \frac{697.650}{7.776} + \frac{3.281.250}{187 * 7.776} = \frac{133.714.800}{1.454.112} = \frac{619.175}{6.732} = 91\frac{6.563}{6.732}$$

As is usual in Pisano's work, the author presented more than one method for his application for the different problems that he dealt with in *Liber abbaci*. In the case of *On a Man Who Invests for Interest Without Notice*, in addition to the one described (*travellers method*), he presented an alternative solution, which he called the *reverse method*. Below is the way in which Pisano presented this methodology:

> In another way, you find this same quantity by the reverse method. For example, the dividend of the fifth year, namely the 30, you add to the $\frac{115}{187}$ 5; there will be $\frac{115}{187}$ 35 that you multiply by 5, and you divided by 6. Because he makes 6 from 5 in each year, the quotient will be $\frac{6\ 11}{11\ 17}$ 29. And this is because that which will remain he will hold in the house 4 years. To this you add the dividend of the fourth year; there will be $\frac{6\ 11}{11\ 17}$ 59 that you multiply by the 5, and divide by the 6; $\frac{5\ 12}{11\ 17}$ 49 will be the quotient; and this much will remain for him as he holds it in the house the third year. And you add this to the 30, namely to the dividend of the third year; there will be $\frac{5\ 12}{11\ 17}$ 79 that you multiply by the 5, and divide by the 6; $\frac{6\ 7}{11\ 17}$ 66 will be the quotient, and this much remains for him after he held it in the house the second year. To this you add the 30, namely the dividend of the second year; there will be $\frac{6\ 7}{11\ 17}$ 96 that you multiply by the 5 and divide by the 6; $\frac{1\ 3\ 6}{6\ 11\ 17}$ 80 will be the quotient, and this much remains for him as he held it in the house one year. And you add it to the dividend of the first year; there will be $\frac{1\ 3\ 6}{6\ 11\ 17}$ 110 that you multiply by the 5 and divide by the 6; $\frac{5\ 1\ 6\ 1\ 6}{6\ 6\ 11\ 17}$ 91 will be the quotient, that is 91 pounds, 19 soldi, and $\frac{1\ 6\ 10}{3\ 11\ 17}$ 5 denari, and this much he held in the house, as we found above by another method. (Sigler, 2003, pp. 388-389)

In the *reverse method*, Pisano started by determining the present value (in year 5) of the cash flow that is in year 5 plus 70 days (1870 days). Once this value was determined, he added it to the cash flow of year 5, that is, 30 denari, and that value was discounted from year 4. Next, to that value he added the cash flow of year 4 and discounted it from year 3, and so on, until reaching to present time.

Then application of the *reverse method* is presented in detail below. To begin, it is necessary to determine the present value, in year 5, of the cash flow that is in year 5 plus 70 days. For this, it is also necessary to establish the appropriate discount rate for a period of 70 days. As previously mentioned, the value of the dividends that the investor would receive at year 5 plus 70 days would amount to $5\frac{5}{6}$ denari, which would be applied to a discount factor equal to $\frac{180}{187}$. This means that the previous dividends (located in year 5 plus 70 days) are equivalent to a dividend in year 5 of $5\frac{115}{187}$ denari. Given the above, it turns out that in year 5 there are two cash flows. The first one, corresponds to the dividends of the investment, which, as indicated, amounts to 30 denari. The second one, corresponds to the present value (in year 5) of the dividends that the investor received in year 5 plus 70 days, that is $5\frac{115}{187}$. In this way, the total value of the cash flow in year 5 is about 35 $5\frac{115}{187}$ denari. As a next step, the author discounted from year 4, the sum of 35 $5\frac{115}{187}$, for which he applied a discount factor equal to $\frac{5}{6}$. In this way, he determined that the present value, in year 4, of the cash flows located in year 5 $\left(35\frac{115}{187}\right)$, amounted to $29\frac{127}{187}$ denari. Then, Pisano proceeded with the discount of cash flows from year 4 to year 3, with the same factor or discount equal to $\frac{5}{6}$. The value of the cash flow to be deducted amounts to $59\frac{127}{187}$ denari, which corresponds to the original cash flow of year 4, about 30 denari, to which the present value of the discounted cash flows was added to year 4, which, as noted, amounted to $29\frac{127}{187}$. By discounting the prior cash flow from year 3, it results in a value of $49\,\frac{137}{187}$ denari. In this state, it continues to discount the cash flow from year 3 to year 2. The cash flow to be deducted amounts to $79\,\frac{137}{187}$, which results from adding the value of $49\,\frac{137}{187}$ denari to the dividend of 30 denari, which is in year 3. When applying the discount factor of $\frac{5}{6}$ on the previous value, it results in a present value, at year 2, of $66\,\frac{83}{187}$ denari. Then, the discount of the cash flows located in year 2 to year 1 is proceeded. The discount rate remains the same, $\frac{5}{6}$. The value to be deducted amounts to $96\,\frac{83}{187}$ denari, which results from adding to the value of $66\,\frac{83}{187}$ denari the dividend of that year, which as is known, amounts to 30 denari. When applying the discount favor, it is observed that the present value in year 1, amounts to $80\,\frac{415}{1122}$ denari. Finally, it is required to discount the cash flows located in year 1, which amount to $110\,\frac{415}{1122}$ denari, to the present time (year 0). This value is the result of adding the dividend of 30 to the present value of future cash flows, which, as indicated, amounts to $80\,\frac{415}{1122}$. Then, after applying the discount factor of $\frac{5}{6}$ to the previous sum, it is found that the present value (in year 0) amounts to $91\,\frac{6.563}{6.732}$ denari. In summary, the value that is necessary to invest for a term of 5 years and 70 days, with annual payments of dividends equal to 30, with an interest of 20 per cent, amounts to $91\,\frac{6.563}{6.732}$ denari.

2.3. Present value concept

Near the end of Chapter 12 Section VI, there is the exercise that made Pisano worthy to occupy not only the base of the tree of financial economics (Barone, 2008), but also to be recognized as the precursor of the present value concept as a capital budgeting criteria (Rubinstein, 2006). In *Liber abbaci*, that exercise is identified as *On a Soldier Receiving Three Hundred Bezants for His Fief*.

In this problem, Pisano presented the case of a soldier whom the king recognizes an annual remuneration equal to 300 bezants for his service. That remuneration is paid to him in four identical quarterly installments of 75 bezants each. The regularity of the payment will change. Instead of being quarterly, the payment will change to an annual payment at the end of the fourth quarter. Under these circumstances, the soldier requests compensation in case of accepting the payment on an annual basis because he would lose the possibility of investing the monies he receives quarterly, which the soldier can invest at a rate of 6 per cent quarterly. Next, the text of the proposed exercise:

> A certain soldier because of his fief received from a certain king 300 bezants each year, and it is satisfied in IIII payments, and in each payment he takes 75 bezants; this is a payment for three months which by necessity is collected together; he asks for a certain compensation in order to accommodate himself for interest because he accepts the 300 bezants instead of the 75 bezants of each payment, namely from payment to payment, of the capital and profit. Voluntarily acquiescing to this he invests the bezants at a profit of two bezants per hundred in each month. It is sought how many bezants he makes in his investment. (Sigler, 2003, p. 392)

For the development of this exercise Pisano applied the *travellers* technique. According to the example, the investment of 100 bezants, produce in a period of three months, 6 bezants, so it is equivalent to say that 100 bezants today are equal to 106 bezants in three months. In this way, 100 bezants become 106, or what would be equal to 50 bezants invested today in three months would become 53 bezants. In other words, the quarterly discount factor is equal to $\frac{50}{53}$. Returning to Pisano's example, there are four quarterly cash flows, each one equal to 75 bezants, located at the end of the quarters one to four, for which it is necessary to find their equivalence at the present moment. In this way, the cash flow located in the first quarter is discounted a quarter with a factor equal to $\frac{50}{53}$; the second cash flow is discounted two quarters, with a factor equal to $\frac{50*50}{53*53}$, the third cash flow, is deducted three quarters, with a factor equal to $\frac{50*50*50}{53*53*53}$, and the fourth and last cash flow is discounted four quarters, with a factor equal to $\frac{50*50*50*50}{53*53*53*53}$. In this way, the present value of the four equal quarterly cash flows of 75 bezants, with a remuneration rate of 6 per cent per quarter, amounts to 259 $\frac{6.966.671}{7.890.481}$ bezants. The process is detailed below:

$$PV = 75 * \left[\frac{50}{53} + \frac{50*50}{53*53} + \frac{50*50*50}{53*53*53} + \frac{50*50*50*50}{53*53*53*53}\right]$$

$$PV = 75 * \left[\frac{50*53*53*53 + 50*50*53*53 + 50*50*50*53 + 50*50*50*50}{53*53*53*53}\right]$$

$$PV = 75 * \left[\frac{7.443.850 + 7.022.500 + 6.625.000 + 6.250.000}{7.890.481}\right] = 259\frac{6.966.671}{7.890.481}$$

In Pisano's words, the solution to this problem:

> First indeed you strive to reduce this problem to the method of trips, and it is reduced thus; because in each month the profit from the 100 bezants is 2 bezants the profit for the one hundred is 6 bezants in the three months, namely at the time of each payment; therefore from each payment of 100 bezants is made 106, that is 53 is made from 50, and because there are IIII payments, IIII trips are similarly carried, and because the payment is 75 bezants, this is had for the expense of each trip. Next because 53 is made from 50, you put $\frac{50}{53}$ four times for the four payments, thus, $\frac{50}{53}\ \frac{50}{53}\ \frac{50}{53}\ \frac{50}{53}$, and you multiply the 50 that is over the first fraction by the 53 that is under the second, and by the 53 that is under the third, and by the 53 that is under the fourth; there will be 7443850. Also you multiply the same 50 by the 50 of the second fraction, and by the 53 of the third, and the 53 of the fourth; there will be 7022500. Again you multiply the first 50 by the second, and by the third; there will be 125000, and you multiply by the 53 that is under the fourth fraction; there will be 6625000. Again you multiply the 50 by the 50, and by the 50, and by the 50, namely those that are over the fractions; there will be 6250000 that you add to the other three just found numbers; there will be 27341350 that you multiply by the 75; there will be 2050601250 that you divide with the $\frac{1\ 0\ 0\ 0}{53\ 53\ 53\ 53}$; the quotient will be $\frac{33\ 6\ 42\ 46}{53\ 53\ 53\ 53}$ 259, and this is the amount of bezants that he makes in his investment. (Sigler, 2003, p. 392).

3. Fibonacci, from anonymity to fame, an eight-century journey

In this section there will be a brief review of some facts that allowed the work of Pisano and his masterpiece *Liber abbaci*, to be rescued.

In the first place, Fra Luca Bartolomeo de Pacioli (ca 1445-1517), an Italian mathematician, Franciscan friar who published his most remembered work

Summa de Arithmetica, Geometria, Proportioni et Proportionalitá (Everything about Arithmetic, Geometry, and Proportions) in Venice in 1494. This work contains a section identified as *Particularis de Computis et Scripturs* (Details of Computation and Recording) (Pacioli, 2010) that deals with the fundamentals of double-entry bookkeeping, which is why Pacioli is now recognized as the *father of accounting and bookkeeping*. For Rubinstein (2006) and Barone (2008), the importance of Pacioli in the history of financial economics is related to the fact that he included the *problem of the points*, also known as the *unfinished game* and despite presenting an incorrect solution, it would later give rise to the development of modern probability (Devlin, 2008). For the case at hand, according to Devlin (2017), Pietro Cossali (1748-1815), Italian mathematician was the one who at the end of the eighteenth-century rediscovered Pisano's work. There are some who accuse Pacioli of plagiarizing[2] (Ciocci, 2017), however, his rediscovery by Cossali four centuries later came about thanks to an appointment made by Pacioli, in his publication. According to Ciocci (2017, p. 107), Pacioli mentioned the contributions of, for example "Euclides, Boecio, Leonardo Pisano, Giordano Nemorario, Biaio Pelacani da Parma, Johannes de Sacrobosco and Prosdocimo de Beldemandis" throughout his work. Regarding Pisano, Ciocci (2017) points out that Pacioli quotes him on nine occasions, even pointing out that the main source of part of his work, was Pisano's work. When reading Pacioli's work, Cossali's[3] observed the next mention: "And since we follow for the most part Leonardo Pisano, I intended to clarify now that any enunciation mentioned without the name of the author is to be attributed to Leonardo" (Devlin, 2017, p. 23). After this new respect for Pisano's work, Giovanni Battista Guglielmini (1763-1817), Italian physicist, published *Elogio di Leonardo Pisano* (In Praise of Leonardo Pisano) in 1812 (Guglielmini, 1812).

As Devlin (2011, 2017) indicated, the Italian historian Guillaume Libri (1803-1869) was the first person to refer to Leonardo Pisano as Fibonacci. Libri published a four volume *Historie des Sciences Mathématiques en Italie, depuis*

2 In Ciocci (2017) review Part 1. No. 7. Luca Pacioli y Piero de la Francesca.

3 In the second volumen of Cossali's work entitled *Origine, transporto in Italia, primi progressi in essa dell'algebra. Storia critica di nuove disquisizioni analitiche e metafisiche* (Origin of algebra as it was brought to Italy and its initial advances thereat) is found in the following terms Pacioli's reference about Pisano: "*E perche noi seguitiamo perla magior parte Lionardo Pisano io intendo dechisrire quando si porra aleuna proposta senza autore quella sia di detto Lionardo*" (Cossali, 1799, p. 25).

la Renaissance des Lettres jusqu'a la fin du dix-septième siècle (History of the Mathematical Sciences in Italy from the Renaissance of Literature to the seventeenth Century) (Libri, 1838a; Libri, 1838b; Libri, 1840; Libri, 1841) between 1838 and 1841 in French. In the second volume, when Libri (1838b) refers for first time to Leonardo Pisano he does it as Leonardo Fibonacci and then simply as Fibonacci. In a footnote, he indicated that Fibonacci is a contraction of *filius* Bonacci, "a Latin phrase that translates literally as 'son of Bonacci'. But Bonacci was not his father's name, so we should perhaps translate the phrase as 'of the Bonacci family" (Devlin, 2011, p. 13). However, in Cossali's publication, it is noted that on the last page of second volume, he used the nickname Fibonacci to refer to Pisano (Cossali, 1799)[4]. Considering the above, the nickname of Fibonacci should be attributed to Cossali who used in 1799 and not to Libri who used it later, in 1838.

With a renewed interest in Pisano's life and work, Boncompagni published several works related to Pisano. For example, in 1852 he published *Della vita e delle opera di Leonardo Pisano matematico del secolo decimoterzo* (Of the life and work of Leonardo Pisano mathematician of the thirteenth century) (Boncompagni, 1852). Then he published *Scritti di Leonardo Pisano* (Writings by Leonardo Pisano) that consists of two volumes: the first of 1857, corresponds to the first reprint, after six centuries of *Liber abbaci* (Boncompagni, 1857), and the second of 1862, corresponds to *Practica Geometriae ed Opusculi* (Practical Geometry) (Boncompagni, 1862). It is important to mention that the reprint of Liber abbaci was made in its original language, Latin.

Later, during the festivities of the patron saint of Pisa, San Ranieri (Luminara of San Ranieri), the town of Pisa inaugurated on June 18, 1863 a marble statue of Pisano, located in the Camposanto of the city. The statue was sculpted by Giovanni Paganucci (ca 1829-1888) and Professor Francesco Buonamici (1832-1921), offered the speech for the occasion (Buonamici, 1863). According

4 The text of Cossali's page note: "Nel corso dell'Opera ho chiamato il benemerito Leonardo di Pisa, Leonardo Bonacci, laddove da altri fu detto Leonardo Fibonacci accozzando la prima sillaba *Fi* di *filius* al paterno nome Bonacci. Io ho stimato di volger questo a cognomen, come assai volte si è fatto. A taluno sarabbe forse più piacciuto il dire Leonardo di Bonacci" (Cossali, 1799, p. 496). Traduction: During the Opera I called the meritorious Leonardo of Pisa, who was called Leonardo Fibonacci by others, by sipling the first syllable Fi di Fillius to the paternal name Bonacci. I have estimated to turn this into a cognomen [nickname], as has often been done. Perhaps someone like Leonardo da Bonacci would have liked more.

to Devlin (2017), when Baron Bettino Ricasoli (1809-1880), politician of Florence, Italy, was in charge of the provisional government of Tuscany, as prime minister, by Decree of September 23, 1859, he decided that the state of Tuscany would finance the elaboration of three statues for the same number of towns in the Italian Tuscany: Pisa, Lucca and Siena. The purpose of each statue "was to commemorate an important local person; the statue for Pisa was to be of Fibonacci. The decree cited him as 'the initiator of algebraic studies in Europe'" (Devlin, 2017, p. 52).

In Buonamici's (1863) speech, he highlighted the commercial vocation of the city of Pisa, which took it for example to ports in Africa, and therefore was thusly influenced by Hindu and Arab culture. From them, Pisano learned the art of numbers that the Arabs received from the Hindus. Pisano showed Europe that the Arabic method of numbering and calculation was perfect and allowed for generating a unique way of expressing ideas. Pisano introduced the nine Arab figures and zero ignored by the Greeks and Romans and fixed the value of the position and deepened the many applications of this new science in commercial life. In another part of his speech, Buonamici rejected the idea that when Pisano returned to Pisa he was not well received and that he was treated with contempt and was called *Bigollo*[5] or *Bigollossu*, saying he did not know the trade of a merchant. Buonamici suggested that Pisano was called that way, perhaps because of his long stay in Bugia or because at that time, the term biglosus was used to call those who had familiarity or knowledge of two languages. Anyway, it was never an insult. Additionally, Buonamici pointed out that the government of the Republic of Pisa hired him to help public officials in the calculation, estimation and numbers and therefore assigned him a monthly salary of twenty liras. Later, Pisano was introduced to the Emperor Frederik II, to whom he dedicated two of his works: *Liber quadratorum* (The Book of Squares) and *Flos super solutioinibus quarundam quaestionum* (Flos) (Buonamici, 1863). In short, Pisano was appreciated by his fellow citizens.

The text of the decree with which the government of Pisa, in 1241, hired the services of Pisano is presented upon continuation:

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5 According to Devlin (2011, p. 13), "a further name Leonardo occasionally used to refer to himself was 'Bigollo', a Tuscan dialect term sometimes used to refer to a traveler, but that meaning may be a coincidence, (In some old dialects the word also meant 'blockhead', but since Leonardo used the term himself, that surely was not his intended meaning)".

In consideration of the honor brought to the city and its citizens and their betterment by the teaching and zealous cooperation of that discreet and learned man, Master Leonardo Bigolli, as well as by his regular patriotic efforts in civic and patriotic affairs, the Pisan Commune and its Officials in certain right and conscious of our prerogative to make recompense for work that he performed in heeding and consolidating the efforts and affairs already mentioned confer upon this same Leonardo so meritorious of our love and appreciation an annual salary or reward from the Commune of 20 free denari and the usual accompaniments. This we affirm with the present statement. (Devlin, 2011, p. 149)

Perhaps one of the best-known exercises in *Liber abbaci*, which would give Pisano recognition, is related with a series that represents the growth of a rabbit population. In that series, a number is the result of the sum of the two previous numbers, starting with 0 and 1. Next, the text and the solution to that problem as given by Pisano:

How Many Pairs of Rabbits Are Created by One Pair in One Year

A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also. Because the abovewritten pair in the first month bore, you will double it; there will be two pairs in one month. One of these, namely the first, bears in the second month, and thus there are in the second month 3 pairs; of these in one month two are pregnant, and in the third month 2 pairs of rabbits are born, and thus there are 5 pairs in the month; in this month 3 pairs are pregnant, and in the fourth month there are 8 pairs, of which 5 pairs bear another 5 pairs; these are added to the 8 pairs making 13 pairs in the fifth month; these 5 pairs that are born in this month do not mate in this month, but another 8 are pregnant, and thus there are in the sixth month 21 pairs; to these are added the 13 pairs that there are born in the seventh month; there will be 34 pairs in this month; to this are added the 21 pairs that are born in the eight month; there will be 55 pairs in this month; to this are added the 34 pairs that are born in the ninth month; there will be 89 pairs in this month; to these are added again the 55 pairs that are born in the tenth month; there will be 144 pairs in this month; to these are added again the 89 pairs that are born in the eleventh month; there will be 233 pairs in this month. To these are still added the 144 pairs that are born in the last month; there will be 377 pairs, and this many pairs are produced from the abovewriten pair in the mentioned place at the end of the one year.

You can indeed see in the margin hoe we operated, namely that we added the first number to the second, namely 1 to the 2, and the second to the third, and the third to the fourth, and the fourth to the fifth, and thus one after another until we added the tenth to the eleventh, namely the 144 to the 233, and we had the abovewritten sum of rabbits, namely 377, and thus you can in order find it for an unending number of months. (Sigler, 2003, pp. 404-405)

According to Devlin (2011, 2017), in the 1870s, the French mathematician Edouard Lucas (1842-1891), called the previous series *Fibonacci numbers*.

Because of this series, Pisano is highly recognized. Although Devlin did not mention exactly in which publication Lucas identified that series as *Fibonacci numbers* also known as *Fibonacci sequence*, it is possible that he refers to the work that Lucas published in 1877 under the title *Recherches sur plusieurs ouvrages de Léonard de Pise et sur diverses questions d'arithmétique supériure* (Research on several works by Leonardo of Pisa and the various questions of arithmetic superior) (Lucas, 1877).

As Rubinstein (2006) points out in the case of present value there is a case of Stigler's law of eponymy because the present value concept is credited to Fisher and not to Pisano, something similar happens with *Fibonacci numbers* that are credited to Pisano and not to its creator. Devlin (2011, 2017) indicates that the first record of the rabbit problem "appeared, it seems, in the *Chandahshastra* (The art of prosody) written by the Sanskrit grammarian Pingala sometime between 450 and 250 BCE" (Devlin, 2011, p. 145).

Although Pisano and *Liber abbaci* disappeared for about six centuries, it cannot be said that his work was lost and had no impact. For example, Devlin (2011) showed that there was later an explosion of texts, written in Italian, directed at a local audience, especially commercial community, known as *libri d'abbaco* (abacus book) or *trattati d'abaco* (abacus tracts). These new books were much shorter than *Liber abbaci*. Researchers such as Gino Arrighi (1906-2001), Italian mathematician who specialized in the mathematics of the Middle Ages and Warren van Egmond, who published in 1980, Practical Mathematics in the Italian Renaissance: A Catalog of Italian Abbacus Manuscripts and Printed Books to 1600, included more than 250 Italian abacus published until 1600. Just as there was a notable increase in the production of abacus books, there was also an important boom in arithmetic schools, in which students learned about the Hindu-Arabic number system. Devlin's (2011) presented the detail of what is considered the oldest known syllabus that comes from the school of Cristofano di Gherardo di Dino, who taught in the Italian city of Pisa, in 1442. The syllabus:

> This is the way of teaching the abacus in Pisa, from he beginning to the end of the students' learning period, as we will say.
>
> 1. At first, when the boy begins school, he is taught how to make figures, that is 9, 8, 7, 6, 5, 4, 3, 2, 1;
> 2. Then he is taught how to keep numbers in his hands, that is his left hand units and tens and in his right hand hundreds and thousands;
> 3. Then to draw numbers on tables: that is of two figures what it means, and then three figures, four figures and son on. Then how to keep them in one's hand.

4. Then one explains the tables of multiplication. One draws it on the table, starting from one times one until ten times ten one hundred, and students learn it very well by heart.
5. Then one teaches how to make divisions;
6. Then how to multiply fractions;
7. Then how to sum fractions;
8. The how to divide [fractions];
9. The how to accrue simple interest and the 'new year's merit';
10. The how to measure lands or how to square a number;
11. Then how to make simple discounts and new year's discounts;
12. Then how to calculate the ounces of silver;
13. Then the melting of silver;
14. Then one makes the comparison between the two amounts;
15. And note that to make the above-mentioned calculations, students are o use pencils according to their level. And sometimes have them sum with their hands, or else on the blackboard; occasionally give them some extraordinary homework, according to the teacher's will.
16. Please, note also this general rule: every evening give them homework for the following day according to their level. And, in case of days of rest, homework is to be doubled. (Devlin, 2011, pp. 109-110)

4. Conclusions

This article was able to verify that thanks to the careful work of a series of curious researchers of the history of mathematics, for example, Cossali (1797), Cossali (1799), Guglielmini (1812), Boncompagni (1852), Boncompagni (1857, 1862), it was possible to rediscover the work of Leonardo Pisano at the end of the seventeenth century and throughout the nineteenth century, especially his masterpiece, *Liber abbaci* but also other works that he developed throughout his life. However, for this discovery to take place, it was necessary for Cossali (1797) to notice in the work of Luca Pacioli (*Summa de Arithmetica, Geometria, Proportion et Proportionalitá*), a reference to Pisano's work, a situation that shows the relevance and importance of citing the works that are consulted.

From the second half of the twentieth century, Arrighi and Egmond's research allowed us to rediscover the boom that was presented after the publication of *Liber abbaci* (first edition in 1202 and second edition in 1228), first of a series of more short books than *Liber abbaci*, written in Italian, usually addressed to the local merchants' community, and second, to the emergence of schools in which children learned the new numbering system with applications to commercial life. The above shows the positive impact that Pisano's work had at the time.

In recent times, it is Goetzmann's (2004) work, which, when reviewing Pisano's *Liber abbaci* in detail, discovered the important and novel financial applications dating from the beginning of the thirteenth century, so much so that his article identifies it as a financial revolution. In this way, Goetzmann's discoveries made it possible to place the origin of net present value as a capital budgeting technique towards the beginning of the thirteenth century instead of the beginning of the twentieth century, as a consequence of Fisher's work (1896, 1906, 1930).

From the publications consulted for the development of this document, it was evident that the nickname of Fibonacci, by which Pisano is currently recognized, is not due to Libri (1838b) but rather to Cossali (1799) who apparently used it for the first time in 1799, pointing out in a note on the last page of that publication, the origin of Fibonacci's nickname. In addition, this research rescued the speech that Professor Buonamici (1863) offered at the inauguration of the Fibonacci's statute in the Camposanto of his hometown, Pisa, on the occasion of the festivities of the saint patriot of Pisa, San Ranieri, in June 1863.

References

Afeera Mubashar, Y. B. (2019). Capital budgeting decision-making practices: evidence from Pakistan. *Journal of Advances in Management Research, 16*(2), 142-167. doi:https://doi.org/10.1108/JAMR-07-2018-0055

Andor, G., Mohanty, S., & Toth, T. (June de 2015). Capital budgeting practices: A survey of central and eastern european firms. *Emerging Markets Review, 23*, 148-72. doi:http://dx.doi.org/10.1016/j.ememar.2015.04.002

Barone, L. (2008). The tree of financial economics. *Rivista di Storia Economica, 24*(3), 373-86. Recuperado de https://ssrn.com/abstract=1123484

Bennouna, K., Meredith, G., & Marchant, T. (2010). Improved capital budgeting decision making: Evidence from Canada. *Management Decision, 48*(2), 225-47. Recuperado de www.emeraldinsight.com/0025-1747.htm

Boncompagni, B. (1852). *Della vita e delle opere di Leonardo Pisano matematico del secolo decimoterzo*. Roma: Tipografia Delle Belle Arti. Recuperado de https://ia800203.us.archive.org/26/items/bub_gb_FoyNx40Ih0gC/bub_gb_FoyNx40Ih0gC.pdf

Boncompagni, B. (1857). *Scritti di Leonardo Pisano - Liber Abbaci*. Roma. Recuperado de https://archive.org/stream/bub_gb_G4IL1D5PUsoC#page/n5/mode/2up

Boncompagni, B. (1862). *Scriti di Leonardo Pisano - La Practica Geometriae ed Opuscoli* (vol. 2). Roma. Recuperado de https://archive.org/details/bub_gb_JYb-VYM12ocC/page/n3

Brunzell, T., Liljeblom, E., & Vaihekoski, M. (March de 2013). Determinants of capital budgeting methods and hurdle rates in Nordic firms. *Accounting & Finance, 53*(1), 85-110. doi:https://doi.org/10.1111/j.1467-629X.2011.00462.x

Buonamici, F. (1863). *Per la inaugurazione nel camposanto di Pisa della statua di Leonardo Fibonacci*. Pisa. Recuperado de https://archive.org/details/perlainaugurazio00buon

Ciocci, A. (2017). *Luca Pacioli. La vida y las obras*. Biblioteca Centro Studi Mario Pancrazi. Cuadernos 15. Recuperado de https://www.aeca.es/old/new/2017/argante_ciocci_pacioli.pdf

Cossali, P. (1797). *Origine, Transporto in Italia, primi progressi in essa dell'Algebra. Storia Crítica di Nuove Dsquisizioni Analitiche e Metafisiche Arricchita* (vol. 1). Italia: Dalla Reale Tipografia. Recuperado de https://archive.org/details/bub_gb_4zPwM_bjgXwC

Cossali, P. (1799). *Origine, Transporto in Italia, primir progressi in essa Dell'Algebra. Storia Critica Di Nuove Disquisizioni Analitiche e Metafidiche Arricchita* (vol. 2). Italia: Dalla Reale Tipografia. Recuperado de https://archive.org/details/bub_gb_cu1eknYNEPIC

Devlin, K. (2008). *The Unfinished Game. Pasacl, Fermat and the Seventeenth-Century Letter that Made the World Modern*. New York: Basic Books.

Devlin, K. (2011). *The Man of Numbers. Fobinacci's Arithmetic Revolution*. New York: Walker & Company.

Devlin, K. (2017). *Finding Fibonacci. The Quest to Rediscover the Forgotten Mathemtical Genius Who Changed the World*. New Jersey: Princeton University Press.

Fisher, I. (August de 1896). Appreciation and Interest. *American Economic Association, XI*(4), 331-442.

Fisher, I. (1906). *The Nature of Capital and Income*. New York: The MacMillan Company.

Fisher, I. (1907). *The Rate of Interest. Its Nature, Determination and Relation to Economic Phenomena*. New York: The MacMillan Company.

Fisher, I. (1930). *The Theory of Interest. As Determined by Impatience to Spend Income and Opportunity to Invest It*. New York: The MacMillan Company.

Goetzmann, W. N. (2004). *Fibonacci and the Financial Revolution*. Working Paper 10352, National Bureau of Economics Research, NBER, Cambridge, MA. Recuperado de http://www.nber.org/papers/w10352.pdf

Guglielmini, G. B. (1812). *Elogio di Lionardo Pisano recitato nella grand'aula della regia Università di Bologna ne giorno XII novembre MDCCCXII dal professore G. B. Guglielmini*. Bolonia: Giuseppe Lucchesini. Recuperado de https://archive.org/stream/bub_gb_kmNnLZenX6kC#page/n1/mode/2up

Libri, G. (1838a). *Historie des Sciences Mathématiques en Italia, depues la Renaissance des lettres jusqu'a la fin du dix-septiéme siécle* (vol. 1). Paris: Jules Renouard et Cie, Libraires, Francia. Recuperado de https://ia801209.us.archive.org/19/items/bub_gb_CF_DvuSzy8cC/bub_gb_CF_DvuSzy8cC.pdf

Libri, G. (1838b). *Historie des Sciences Mathématiques en Italie, depuis la Renaissance des Lettres jusqu'a la fin du dex-septiéme siecle* (vol. 2). Paris. Recuperado de https://ia800406.us.archive.org/29/items/bub_gb_4lLOM16x0FgC/bub_gb_4lLOM16x0FgC.pdf

Libri, G. (1840). *Historie des Sciences Mathématiques en Italie, depuis la reinassance des lettres jusqu'a la fin du dix-septiéme sciele*. Paris. Recuperado de https://ia802701.us.archive.org/21/items/histoiredesscie01librgoog/histoiredesscie01librgoog.pdf

Libri, G. (1841). *Historie des Sciences Mathématiques en Italiem despues la reinassance des lettres jusquá la fin du dix-setiéme siecle* (vol. 4). Paris. Recuperado de https://ia800205.us.archive.org/14/items/histoiredesscie02librgoog/histoiredesscie02librgoog.pdf

Lucas, E. (1877). *Recherches sur plusieurs ouvrages de Léonard de Pise et sur diverses questions d'arithmétique supériure*. Paris. Recuperado de http://www.math.utah.edu/~beebe/software/java/fibonacci/lucas_book_leonard_de_pise.pdf

Pacioli, L. (2010). *The Rules of Double-Entry Bookkeeping*. Lexington: IICA Publications.

Ross, S. A., Westerfield, R. W., & Jaffe, J. (2013). *Corporate Finance*. New York: McGraw-Hill Irwin.

Rubinstein, M. (2003). Great Moment in Financial Economics: I. Present Value. *Journal of Investment Management, 1*(1).

Rubinstein, M. (2006). *A History od the Theory of Investments. My Annotated Bibliography*. New Jersey: John Wiley & Sons.

Sigler, L. (2003). *Fibonacci's Liber Abaci: A Translation into Modern English of Leonardo Pisano's Book of Calculation*. New York: Springer-Verlag.

Souza, P. d., & Lunkes, R. J. (2016). Capital budgeting practices by large Brazilian companies. *Contaduría y Administración, 61*(3), 514-34. doi:https://doi.org/10.1016/j.cya.2016.01.001

Wnuk-Pel, T. (2014). The practice and factor determining the selection of capital budgeting methods - evidence from the field. *Procedia - Social and Behavioral Sciences, 156*, 612-16. doi:https://doi.org/10.1016/j.sbspro.2014.11.250